Centre of Excellence for Biosecurity Risk Analysis

ML vs MRR: Weibull Parameter Estimation for Making Decisions

ACERA Project

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Contents

List of Tables

List of Figures

Nomenclature

Symbol	Meaning
α	The shape parameter for the Weibull distribution.
β	The scale parameter for the Weibull distribution.
cdf	Cumulative density function.
Censoring, Type-1	Observation of all units is terminated after a given amount of time, regardless of how many units have failed.
Censoring, Type-2	Observation of all units is terminated after a given number of units have failed.
$F_X(x)$	The cdf of the random variable X evaluated at observation x.
$f_X(x)$	The pdf of the random variable X evaluated at observation x.
$h_X(t)$	The hazard of X evaluated at t — that is, the probability that an item from random variable X fails at time t, given that it has not failed before time t.
MIAE	The mean integrated absolute error.
MISE	The mean integrated squared error.
ML, MLE	Maximum likelihood (Estimate).
MRR	Median rank regression.
$p_k(X)$	The time at which the hazard of random variable X exceeds 10^{-k}.
pdf	Probability density function.
RMSE	Root mean squared error.
R_k	The relative cumulative failure probability arising from the value of $p_k(t)$ based on estimates of α and β, as opposed to the known values.
$X,\ x$	A random variable representing the failure time and an observation from the random variable.

Executive Summary

Précis

Median rank regression estimation (MRR) showed superior performance compared with *maximum likelihood* estimation (ML) using operationally relevant parameter estimation criteria in a simulation experiment that covered an operationally relevant range of conditions.

Background

Fleet management requires timely and accurate decisions regarding vehicle maintenance. Fleet maintenance scheduling is based on statistical analysis of measured performance characteristics, among other things. The statistical analysis commonly involves the fitting of a Weibull distribution to observed failure times and the calculation of decision-relevant statistics, for example the hazard cutoff, which is the earliest hour at which the predicted hazard of failure exceeds a threshold, for example 10^{-6}.

The observed failure data at the decision time are incomplete (censored) because not all vehicles (or components within vehicles) have failed. Two methods are commonly used for the fitting of the Weibull distribution to the censored data, namely, MRR and ML. These two methods are known to have different relative performance depending on the metric, the censoring type, the number of vehicles, and so on.

Experiment

We performed a suite of simulation experiments to assess the relative performance of MRR and ML under a range of operationally realistic conditions. We compared the performance using a set of performance measures, focusing on the hazard cutoff, which is used for the decision point of when to withdraw the fleet for maintenance.

Results

Focusing on the hazard cutoffs, the variability of the ML and MRR estimates were comparable, however the ML estimates showed considerable positive bias — meaning that the hazard cutoffs were too high, and intervention decisions would come too late — whereas the MRR estimates were negatively biased, meaning that intervention decisions would come too early, on average. Hence the realized risk for a decision made using ML estimates was substantially higher than the risk implied by policy, and that of MRR was lower.

Discussion

We conjecture that MRR performance was better than ML using the risk ratio performance measure because the measure focuses entirely on the left tail of the distribution, and that the error in estimating the shape parameter is, relatively speaking, more influential on the behaviour of the left tail than is the error in estimating the scale parameter. Our results, and the results of previous work, show that MRR is better than ML for estimating the shape, and ML better than MRR for scale. We further conjecture that estimating the parameter values is influenced by the geometry of parameter space for ML and MRR; the former is directly influenced by the censored data, while the latter is indirectly influenced by the censored data.

Recommendations

We recommend that MRR be used if the purpose of estimation is to guide the decision of when to withdraw a fleet for maintenance. We also recommend that further work be undertaken to develop estimators with better performance in the left tail.

1 Introduction

Managing the risk of failure is a key challenge for the management of fleets of vehicles. All vehicles are constructed from components, the failure of which can compromise the safety of the vehicles and therefore the fleet. Typically the maintenance schedule of the fleet is set to handle the component type that has the shortest expected lifespan.

Some component systems are critical, meaning that their failure will incapacitate the vehicle. Other systems have redundancy, so the failure of individual components would be inconvenient but not necessarily incapacitating. Either way, it is preferable to replace the components before they fail. However, replacing components unnecessarily is expensive and would result in unacceptable downtime for the fleet. Therefore the preferred strategy involves replacing the components before the risk of their failure becomes too great.

There are several different ways of representing the risk of running the fleet. We focus on the hazard, that is, the probability of failure within a given unit time given that the component has survived up until the start of the time unit. Here we take the time unit to be an hour. We use as risk-based cutoffs the values 10^{-6} for alert and 10^{-5} for action. That is, we are interested in predicting the first hour for which the hazard of failure of a component in a fleet, h, is greater than 10^{-6} and likewise greater than 10^{-5}.

Predicting the probability of component failure is a very difficult challenge. The original equipment maker (OEM) takes a series of design steps to determine the probability distribution of failures for the component. From this process, the OEM includes safety factors to significantly reduce the possibility of a failure in operation by determining a life-limit for the component. For example, the OEM may define the life-limit of the component as the probability of failure $= 10^{-4}$ at the 90% confidence level.

However, failures do occur in operation, which can effectively invalidate the OEM's predicted life-limit for the component. The OEM uses available failure and censored data to revise the probability of failure of the balance of the components. This in turn enables better management of the safety and availability of the fleet from a set of risk-based cutoffs or decision points.

Making a decision after observing only the earliest failure times means that the failure times of the balance of the vehicles are not observed. In statistical terms, these unobserved values are *censored*, more specifically, *Type-2 right censored*. This means that observation of all units is terminated after a given number of failures (Genschel and Meeker, 2010). In contrast, *Type-1* censoring occurs when observation of all units is terminated after a given amount of time, regardless of how many units have failed. The statistical model needs to take account of the censored data somehow, in order to use the available information as efficiently as possible.

1.1 Background

The two-parameter Weibull probability function is very popular for such applications due to the wide range of shapes that it can take (Figure 1). A brief review of the breadth of its applications follows. This distribution is characterised by the following probability density function (pdf),

$$f(X = x \mid \alpha, \beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)$$

The Weibull distribution is one of the most widely used distributions for the analysis of lifetime and time-to-event data (see, e.g., Ma and Krings, 2008; Genschel and Meeker, 2010). It has been frequently applied in the analysis of survival data in biomedical and ecological science (see, e.g., Ricklefs, 2000; Carroll, 2003) and reliability testing in engineering (see, e.g., Lincoln, 1985; Shokrieh and Rafiee, 2006; Guo et al., 2014). Survival analysis and reliability theory effectively address mathematically similar problems (Ma and Krings, 2008) and the majority of this data type is considered to be adequately modelled by a Weibull distribution (Skinner et al., 2001).

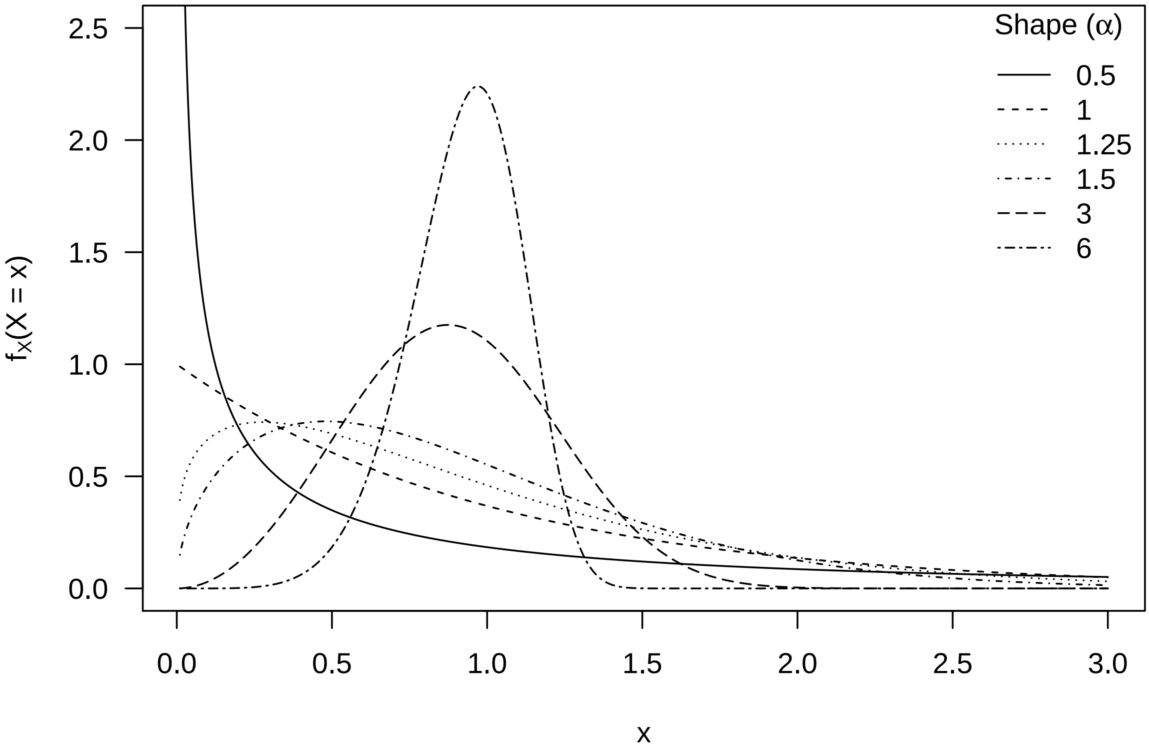

Figure 1: Weibull curves for each of the shape parameters used in the simulations, with scale = 1.

In recent years there has been a greater focus on the Weibull distribution in other disciplines, such as wind energy prediction, due to its flexibility, simplicity and good fit to experimental data (see, e.g., Lun and Lam, 2000; Seguro and Lambert, 2000; Celik, 2006; Nedaei, 2014; Ramos and Iglesias, 2014). Its history of use in disciplines as diverse as weather forecasting (Gorter et al., 2014); particle size distributions (Zobeck et al., 1999); species dispersal (see, e.g., Higgins and Richardson, 1999; Zabel et al., 2014); and the assessment of forest stand structure (Bailey and Dell, 1973; Siipilehto, 2009); highlights its wide applicability.

Recently authors have been applying the Weibull distribution to novel areas of research to assess its efficacy for adequately addressing statistically equivalent time-to-event data. For example, Papadopoulou et al. (2006) provided experimental evidence to support the use of the Weibull function in drug release studies and Grissino-Mayer (1999) tested the Weibull distribution as a possible method to model fire frequency data, concluding it provided a valuable theoretical framework to assess historical variation in fire regimes, while acknowledging that some of the ecological interpretations of the Weibull parameters were complex.

Even in areas of research where conventional models have been used consistently over a long period, authors are demonstrating the Weibull distribution to be an effective alternative, often with greater explanatory power. Watt et al. (2010) found that despite hydrothermal time (HTT) models with normal distribution being extensively used for modelling of seed germination, HTT models with a Weibull distribution more accurately described the right skewed distribution of data and germination time. In another instance, a comparison of Weibull models with Cox proportional hazards models, which make no assumption about the underlying distribution, illustrated that Weibull models may be more informative and of greater value to researchers in the analyses of some clinical trial data (Carroll, 2003).

The Weibull distribution is a continuous probability distribution and the accurate estimation of the shape and scale parameters is fundamental to the distribution features, including the shape of the pdf and the estimated rates of failure or time-to-event. Numerous graphical and analytical methods to estimate Weibull parameters under different censoring regimes have been proposed (Razali et al., 2009) and median rank regression (MRR) and maximum likelihood (ML) are the most frequently cited (Wang, 2004; Genschel and Meeker, 2010). We compare these two parameter estimation algorithms in this report. A brief review of each follows.

1.1.1 Maximum Likelihood Estimation

The goal of maximum likelihood estimation is to identify the parameter estimates that make the data as likely as possible, conditional on the model. Here, the model is the Weibull probability function, and the parameters are called the shape (α) and scale (β). The likelihood is algebraically identical to the joint pdf of the sample of data, however, it is realized as a likelihood by evaluating it conditional on the parameters instead of conditional on the data. For the purposes of our development, imagine that we observe the first k observations and stop observing at time t. We use the joint pdf for the k observed failure times, so we need to find parameter values α and β that satisfy

$$\max_{\alpha,\beta} \prod_{i=1}^{k} \frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x_i}{\beta}\right)^{\alpha}\right).$$

One important attribute of censored data is that the variables for one or more observations are not measured, because the experiment or exercise terminates before the awaited conditions have occurred. In the present case, the example is of a fleet of vehicles that is decommissioned for repair after the failure of a small proportion of them. For all the other vehicles, we know that they will fail at some point, and that they have not failed up until the point at which the observations cease. These observations contribute the reverse of their cumulative density to the likelihood, that is, an amount equal to the probability that they have *not* failed up until the time of observation, as a function of the parameters. So we use the joint survival function for the other $n-k$ observations, evaluated at time t.

$$F(\mathbf{x} \mid \alpha, \beta) = \prod_{i=1}^{n-k} \exp\left(-\left(\frac{t}{\beta}\right)^{\alpha}\right).$$

The contributions to the likelihood for the observed (pdf) and the unobserved (cdf) observations are simply multiplied, as follows.

$$\max_{\alpha,\beta} \prod_{i=1}^{k} \frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x_i}{\beta}\right)^{\alpha}\right) \times \left(\exp\left(-\left(\frac{t}{\beta}\right)^{\alpha}\right)\right)^{n-k}$$

In almost all applications, it is more convenient to maximize the log of the likelihood than the likelihood, as this changes the underlying nature of the joint likelihood from being a product to being a sum. Therefore the maximum likelihood estimates (MLEs) are defined as the parameter values α and β that satisfy

$$\max_{\alpha,\beta} \left\{ k(\log\alpha - \log\beta) + \sum_{i=1}^{k} (\alpha-1) \times (\log x_i - \log\beta) - \left(\frac{x_i}{\beta}\right)^{\alpha} - (n-k) \times \left(\frac{t}{\beta}\right)^{\alpha} \right\} \quad (1)$$

1.1.2 Median Rank Regression

Median rank regression (MRR) takes a different approach to obtaining the parameter estimates. The genesis of MRR is that all data follow a straight line, on average, if plotted appropriately against the quantiles of the population distribution. In the Weibull case, estimation relies upon the following reasoning:

if $x_i \stackrel{d}{=} \text{Weibull}(\alpha, \beta)$,
then $\log x_i$ and $\log(-\log(1 - r_i))$ are (on average) linearly related,

where r_i are smoothed ranks of x_i, and the relationship is a function only of α and β. We remark that this principle is commonly used as the basis of a graphical diagnostic tool for assessing the fidelity of observations to the Weibull distribution.

Two complications are necessary to improve the performance of the estimator, namely correction of the ranks to account for censoring, and an ad-hoc correction that shifts the adjusted ranks to their median values. The algorithm for fitting that we used was as follows, following Genschel and Meeker (2010), see also Benard and Bos-Levenbach (1953).

- Order all times (failure and censored) from lowest to highest.
- Assign increasing ranks $1, \ldots, n$.
- Considering only the ranks of the observed failures, say k_i, $i = 1, \ldots j$,
  - start with $r_0^* = 0$ and
  - iteratively compute $r_i^* = r_{i-1}^* + \frac{n+1-r_{i-1}^*}{n+1-(k_i-1)}$
- Form the linear predictor variable by

$$v_i = \log\left(-\log\left(1 - \frac{r_i^* - 0.3}{n + 0.4}\right)\right)$$

- Construct a least-squares regression model for only the observed failure times, $\log t_i = \mu + \sigma v_i + \epsilon_i$, where ϵ_i are residuals.
- Form the MRR parameter estimates by $\hat{\alpha} = 1/\hat{\sigma}$ and $\hat{\beta} = \exp(\hat{\mu})$.

Note that since we are using only the parameter point estimates from the least-squares model, we don't need to assume anything about the distribution of the residuals or the nature of the variation.

1.1.3 Comparison

Simulation studies that compare the parameter estimates arising from the MRR and ML algorithms are common (for example, Kuchii et al., 1979; Gibbons and Vance, 1981; Khalili and Kromp, 1991; Montanari et al., 1997; Skinner et al., 2001; Hossain and Zimmer, 2003; Razali et al., 2009; Genschel and Meeker, 2010; Olteanu and Freeman, 2010; Chu and Ke, 2012; Teimouri et al., 2013; Guure and Ibrahim, 2013). An exhaustive review is beyond the scope of this report.

Many studies examining the statistical properties of the estimators have found substantial differences in the accuracy and bias of estimators (see, e.g., the review by Genschel and Meeker, 2010), and there is some contention over which performs best. The conflict largely arises due to different conclusions reached by authors who are assessing estimators under different study conditions, including a range of censoring types and levels, in addition to those studies often using different metrics for their assessment criteria (e.g. mean bias, mean square error (MSE) Gibbons and Vance, 1981; Skinner et al., 2001; Hossain and Zimmer, 2003; Razali et al., 2009; Genschel and Meeker, 2010; Chu and Ke, 2012) and mean absolute deviation (see e.g., Genschel and Meeker, 2010; Olteanu and Freeman, 2010), and predicting the future number of failures (Olteanu and Freeman, 2010).

ML is considered by many authors to be more versatile than MRR as it provides consistently good parameter estimates with different censored data including interval and truncated censoring, it can be used with regression modelling, and provides defensible estimates from small sample sizes (Genschel and Meeker, 2010; Olteanu and Freeman, 2010). Genschel and Meeker (2010) acknowledged conflicting opinions on performance of ML and MRR and sought to address these differences by conducting extensive simulation studies. They found the ML performed better than MRR in all but a small fraction of cases, mostly based on the MSE metric.

In contrast, several authors have found that the least squares (LS) method, also known as the rank regression, outperforms ML. This was demonstrated for small sample sizes using sample root mean square errors by Chu and Ke (2012). Hossain and Zimmer (2003) assessed the performance of ML and LS on complete, multiply time censored and Type-2 censored data with a range of sample sizes. While the results were mixed based on the bias and MSE of the estimates, the authors advocated the use of LS if one estimator were to be consistently used.

Often conflicting opinions are compounded by studies finding no clear evidence to justify the use of one estimator over another. Common issues for time-to-failure analyses are the very small numbers of failures and extremely high right censoring imposed upon the data. Olteanu and Freeman (2010) assessed ML and MRR with respect to these factors using different censoring scenarios but in Monte Carlo simulation experiments found neither method clearly outperformed the other overall, and that neither method provides stable estimates for failure points of less than ten and large censoring. The authors based their conclusions on the median absolute deviation (MAD) and a pseudo-MSE type measure. In another study, a simulation comparison of ML and MRR estimates derived for random Type-1 censored data, using MSE and mean percentage error for small, medium and large sample sizes, ML provided better estimates for the scale parameter but LSE on x was slightly higher for the shape parameter, though not so substantially as to advise against using ML (Guure and Ibrahim, 2013).

Genschel and Meeker (2010) provided a small collection of rationales for preferring ML to MRR, namely

1. MRR uses ordinary least squares (OLS) to fit the straight line to the transformed data, but the assumptions under which OLS estimation is known to have good statistical properties are not satisfied, notably constant variance and independent observations; and
2. OLS estimators put large weight on the extreme observations, which in the case of quantile data, have large variance.

However, the possibility that there could be drawbacks in the OLS estimator relative to its optimal application neither invalidates the estimator nor necessarily leads to the preference of ML. It is quite possible that the estimator may perform better than ML for certain applications despite its apparent shortcomings.

On the other hand, the points most commonly cited in favour of ML in the case of the Weibull distribution are asymptotic qualities, which are not necessarily relevant at the small sample sizes that are realized in high-censoring conditions. However, an important advantage of ML is that monotonic functions of MLEs are themselves MLEs of the functions of the parameters. Therefore the attractive qualities of ML carry directly over to such functions of the parameters, which are often of greater interest than the parameters themselves, such as in the case that motivates this study.

2 Materials and Methods

The basis of the project was a sequence of simulation experiments designed to assess the performance of ML and MRR for parameter estimation for the 2-parameter Weibull distribution. The simulation experiments were carried out using the open-source statistical environment, R (R Core Team, 2014). We nominated an operationally useful range of values for the scale parameter, namely 500–50000, and a range of shape values that covered the likely values of interest, namely 0.5–6. We considered only one kind of censoring, namely Type-2 right censoring, because these are the decision conditions under which the parameter estimates are used in this example of fleet management.

2.1 Design

The baseline simulation experiment comprised the following experimental factors:

- four ways of staggering observations;
- four different scales (500, 2500, 5000, 50000);
- eight different shapes (0.5, 0.75, 1, 1.25, 1.5, 3, 4.5, 6);
- five different sample sizes (20, 50, 100, 500, 1000);
- seven different failure rates (1%, 2%, 5%, 20%, 50%, 70%, 90%, 100%); and
- 1000 replications.

For each combination of the experimental factors, we recorded the following statistics:

1. bias of shape and scale, reported as relative bias;
2. MIAE, MISE of pdfs relative to the specified process pdf, so

$$\text{MIAE} = \int_0^\infty |f_X(X = x \mid \hat{\alpha}, \hat{\beta}) - f_X(X = x \mid \alpha, \beta)| dx$$

$$\text{MISE} = \int_0^\infty \Big(f_X(X = x \mid \hat{\alpha}, \hat{\beta}) - f_X(X = x \mid \alpha, \beta) \Big)^2 dx$$

   where f_X is the Weibull pdf, and we approximated the integrals by splitting the true Weibull density into chunks by its quantiles, computing the area within each chunk using the difference of the cdfs evaluated at the top and bottom of each portion, and taking the absolute value of the difference between that area and the area computed using the same chunk locations applied to a Weibull density with the estimated parameters;
3. bias and RMSE of each of four *hazard cutoffs*, here defined as the first hour for which

   (a) $p_6(= \Pr(\text{fail during hour} \mid \text{survival until hour begins})) > 10^{-6}$;

   (b) $p_5(= \Pr(\text{fail during hour} \mid \text{survival until hour begins})) > 10^{-5}$;

   (c) $p_{\bar{6}} > 10^{-6}$; and

   (d) $p_{\bar{5}} > 10^{-5}$.

   Here, the average indicates a cumulative average, that is, p_6 is the first hour at which the failure hazard in the fleet is greater than 10^{-6}, and $p_{\bar{6}}$ is the first hour at which the average failure hazard is greater than 10^{-6}. The true values for comparison are provided in Table 1.
4. risk ratios for the four hazard cutoffs (see Section 2.1.1).

As noted above, there are many different statistical criteria for estimate quality, for example the bias, RMSE, and MAD of the parameter estimates, and the criteria do not always agree on which estimation algorithm is to be preferred. Even then, focusing on the statistical qualities of the parameter estimates is both misleading and constraining.

First, it is misleading in the sense that the parameters are merely the convenient statistical access points of the distribution. Statistical properties of parameters, such as bias and RMSE, do not carry across to non-trivial (meaning, specifically, not linear) functions of the parameters. If the parameter estimates themselves are not of direct interest, such as is the case in the current study, then the statistical qualities of their estimation can be a distraction.

Second, focusing on the statistical properties of the parameters impedes the extension of the simulation exercise to include distributional uncertainty. Although it is convenient to assume the Weibull distribution, for example, for fitting to the data, there is no guarantee that the data themselves follow the Weibull law. If the intent is to use estimators to make decisions

Table 1: True hazard values for all scenarios. p_x refers to the first hour at which the probability of a failure within the fleet is greater than 10^{-x}. $p_{\bar{x}}$ refers to the first hour at which the cumulative average probability of a failure within the fleet is greater than 10^{-x}. Note that all true hazard values are 1 for all values of shape less than or equal to 1.

Shape (α)	Scale (β)	p_6	p_5	$p_{\bar{6}}$	$p_{\bar{5}}$
1.25	500	1	1	1	1
1.50	500	1	1	1	1
3.00	500	6	20	10	34
4.50	500	37	72	56	109
6.00	500	76	121	109	173
1.25	2500	1	1	1	1
1.50	2500	1	1	1	1
3.00	2500	72	228	124	395
4.50	2500	294	567	451	872
6.00	2500	527	836	754	1196
1.25	5000	1	1	1	1
1.50	5000	1	6	1	10
3.00	5000	204	646	353	1120
4.50	5000	716	1384	1100	2130
6.00	5000	1211	1920	1732	2753
1.25	50000	1	1337	1	3340
1.50	50000	56	6043	123	14605
3.00	50000	6462	21206	11211	39945
4.50	50000	13836	27185	21309	44424
6.00	50000	19205	30749	27540	46160

based on estimated quantiles of the distribution, for example, then determining the estimation algorithm that provides estimates that are robust against distribution misspecification may be important. Previous authors have also made this point, for example Higgins and Richardson (1999) highlighted that there is a conflict between the biological and statistical criteria used to choose the most appropriate statistical model, and we quote from Olteanu and Freeman (2010):

> "Even in terms of estimation, there are two important but subtly different questions. The first deals with the quality of the estimated parameters, which is straightforward. The second, which is more subtle, focuses on the risk management issue, which predicts the expected number of failures within some window given the first several failures. For example, the Air Force needs to know the expected number of jet engine failures over the next year, given the first set of failures that have already occurred. Such information is necessary for providing sufficient inventories of spare jet engines in the field."

2.1.1 Risk ratio

We developed a further index of parameter estimate quality, namely, the *risk ratio*, which reports the relative realized risk arising from parameter estimates. The rationale is as follows. If the four hazard cutoffs are to be used as the basis of managing the risk associated with the fleet, then they imply a cumulative risk, say, P_f. That is, deciding to withdraw the fleet for maintenance at a specific hazard cutoff results in a specific cumulative probability of failure, which is equal to the cdf of the failure process, evaluated at the estimated hazard cutoff. The realized cumulative probability of failure for using $\hat{p}_6$ is

$$P_6 = \int_0^{\hat{p}_6} f_X(x \mid \alpha, \beta) dx \tag{2}$$

where $f_X(x \mid \alpha, \beta)$ is the Weibull pdf evaluated at the (unknown) process values α and β . However, the estimated, or intended, risk is based on the estimates $\hat{\alpha}$ and $\hat{\beta}$, that is,

$$\hat{P}_6 = \int_0^{\hat{p}_6} f_X(x \mid \hat{\alpha}, \hat{\beta}) dx \tag{3}$$

We refer to this quantity as the intended risk because it is the risk that the decision-maker is choosing to accept via the nomination of $\hat{p}_6$ as a function of $\hat{\alpha}$ and $\hat{\beta}$.

In a simulation setting, we know both α and β, so we can compute P_x and $\hat{P}_x$ for any estimation algorithm. We report the ratio, $R_x = P_x/\hat{P}_x$, for x in $\{5, 6, \bar{5}, \bar{6}\}$ as a metric of the quality of the parameter estimates in the context of this decision. The results are ratios, so using the arithmetic mean to summarize the simulation results is not appropriate. In order to summarize the simulation results appropriately, we use the geometric mean, and form confidence intervals on the log scale, then transform the estimate and interval limits back to the ratio scale.

If $R_x < 1$ then the estimates lead to conservative decision-making, drawing the fleet in early. If $R_x > 1$ then the estimates lead to anti-conservative decision-making, exposing the decision-maker to greater total risk than intended.

2.2 Staggering entry

In order to provide a realistic picture of the motivating operational conditions under which observations are developed, we needed to provide for the possibility that the start times of the use of the components were heterogeneous. An example of such a situation is the ad-hoc replacement of a failed widget in a vehicle; the replacement widget enjoys a later start time than do its contemporaries. Other management scenarios would lead to different kinds of patterns, for example there may be some vehicles in the fleet that are used more frequently than others, leading them to age more quickly, vehicles might be inspected or maintained in groups, and so on.

A considerable amount of early effort in the project was devoted to developing suitable algorithms for staggering the entries of observations, in order that there be a variety of censoring times. We applied four different staggering algorithms in the study, and we cover their results here and do not mention them further.

1. No staggering of entry times.

2. Replace all censored times with random draws from the same Weibull distribution but at random censoring times, where the latter are drawn by uniformly selecting Weibull quantiles up to and including the quantile corresponding to the censoring time.

3. As per the previous algorithm except instead of drawing up to and including the quantile corresponding to the censoring time, draw up to and including a uniform random point between the quantile corresponding to the censoring time and the endpoint of the distribution.

4. Weibull stagger: all components start at the same time but failed components are replaced using components with identically distributed failure times. Here, depending on the failure count that guides the stopping rule, the majority of the censored observations will be at identical times, and the balance, which are replacement widgets, will be censored earlier in their lifetimes.

The simulation results were comparable for the first and fourth staggering algorithms, which after a moment's reflection is not surprising — the only difference is in the sample size, as we considered each replacement widget to be a new component. Therefore the effective sample size was a function of the stopping rule, and was always higher than that prescribed. However, in any case, the observations were all drawn from the Weibull distribution, so the results would be expected to be very similar.

The simulation results for the second and third algorithms were counter-intuitive, with, for example, bias in the MLE *increasing* with sample size. This odd outcome is a result of a substantial number of the observations no longer following a purely Weibull law, owing to their variable censoring. Furthermore, the hypothetical scenarios that informed these algorithms did not match the likely operational conditions. We do not consider these scenarios further.

Accordingly in the balance of the report we focus on the first algorithm, namely, no staggering.

2.3 Simulation platform correspondence

Early work in the project also focused on ensuring that the outcomes of simulations in R and Mathematica were commensurate. Although the process of inducing the applications to agree was painstaking and time consuming, it was useful in terms of ironing out small inaccuracies in the algorithms. The default arguments for the `optim` function in R did not perform well in for maximizing the log-likelihood of the Weibull distribution in some circumstances. The default algorithm used by Mathematica for resolving ties in sorting data led to incorrect implementation of MRR. The R code used for the simulations is provided in the Appendix of this document. The R code calls the following packages: ggplot2, survival, lattice, and parallel.

3 Results

This report provides full results for a subset of the simulations, namely a base case with no staggering and 100% fails, a censored case with no staggering and 20% fails, and two special cases that are considered to cover operationally realistic scenarios.

3.1 Base Case

The base case was developed to provide a reference point for comparison with previous simulation exercises. The simulation specification for the base case is as follows.

- no staggering;
- four different values of scale (500, 2500, 5000, 50000);
- eight different values of shape (0.5, 0.75, 1, 1.25, 1.5, 3, 4.5, 6);
- five different sample sizes (20, 50, 100, 500, 1000);
- 100% failures (no censoring); and
- 1000 replicates.

Figure 2 shows the simulation results for the bias of the estimate of the shape parameter, α, for both ML and MRR in the base case experiment. The pattern is relatively constant across the range of scale, so results corresponding to only two values of scale are provided. The relative bias in estimates of α from both ML and MRR decreases towards 0 with increasing sample size, and is relatively constant across the range of true shape. The magnitude does not seem to change with true scale. The bias is positive for ML and negative for MRR, and seems to be about twice as large for the former as the latter for small sample sizes, but comparable for sample sizes of 100 or more.

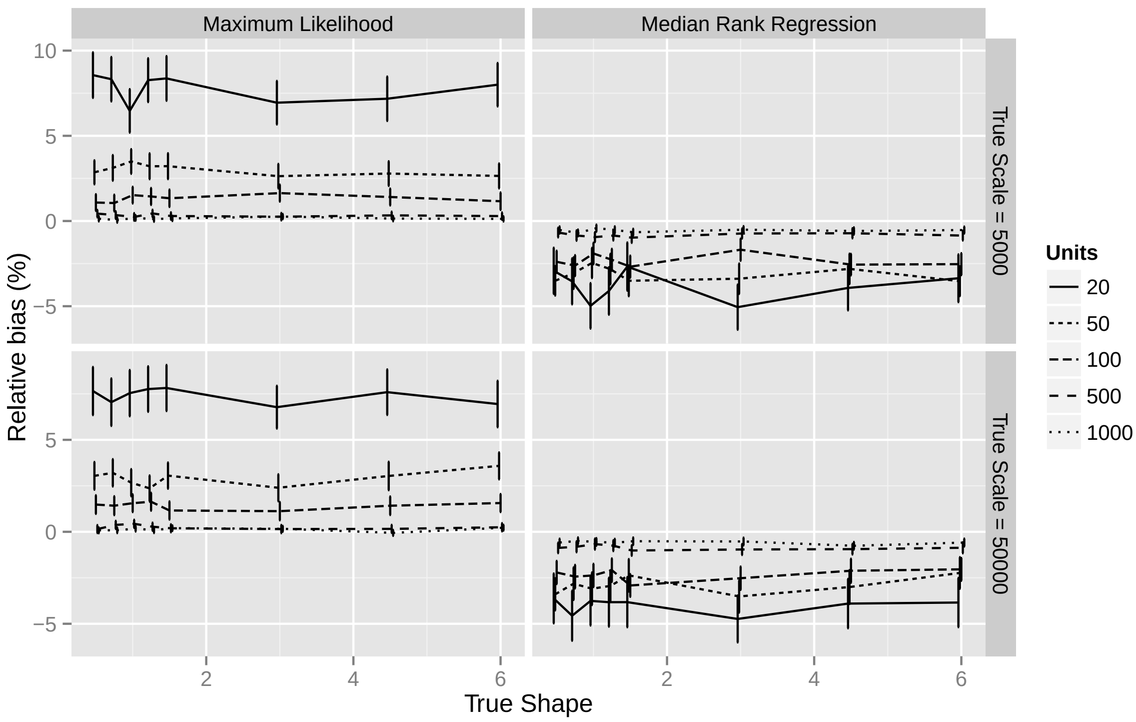

Figure 2: Relative bias of estimate of shape parameter α, 100% fail. Each point is the mean of $(\hat{\alpha} - \alpha)/\alpha \times 100$ from 1000 simulations; the vertical bars are 95% confidence intervals.

Figure 3 shows the simulation results for the bias of the estimate of the scale parameter, β, for both ML and MRR in the base case experiment. The pattern is relatively constant across the range of scale, so results corresponding to only two values of scale are provided. Relative biases of scale for both ML and MRR are largely unaffected by true scale. Bias is positive for ML and reduces very close to zero with increasing sample size and increasing shape. Bias is positive and larger, and decreases to close to zero with increasing sample size and shape for MRR.

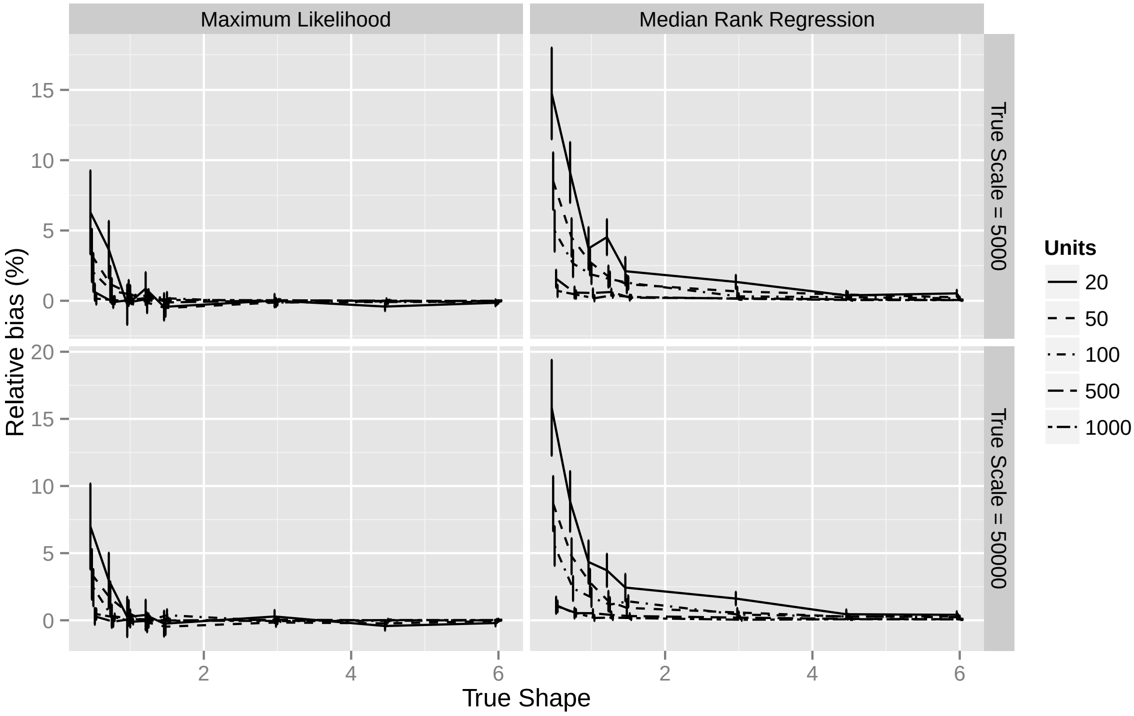

Figure 3: Relative bias of estimate of scale parameter β, 100% fail. Each point is the mean of $(\hat{\beta} - \beta)/\beta \times 100$ from 1000 simulations; the vertical bars are 95% confidence intervals.

Figures 4 and 5 provide the MIAE and MISE of the estimated distributions respectively. MIAE is independent of shape and decreases with scale, reduces in magnitude and variability with increasing sample size, and is slightly lower for ML than MRR. MISE is the same but substantially lower for ML than MRR, and substantially more variable for MRR than ML.

Figure 6 presents the p_6 hazard bias from the simulations. The biases arising from both estimates seem to increase with scale, decrease with sample size, and increase with shape, as do the values themselves (Table 1). The bias seems slightly larger in absolute value for ML estimates than MRR estimates, but starts positive and switches to negative with increasing shape. The results were broadly similar for the bias of the p_5, $p_{\bar{6}}$, and $p_{\bar{5}}$, hazard bias, so the results are not presented here.

Figure 7 presents the p_6 hazard *relative* bias from the simulations. The relative biases arising from both estimates seem to increase with scale, decrease with sample size, and increase initially and then descend with shape. The location of the peak relative to true shape seems to change with true scale. The relative bias seems larger in absolute value for ML estimates than MRR estimates. The results were broadly similar for the bias of the p_5, $p_{\bar{6}}$, and $p_{\bar{5}}$, hazard bias, so the results are not presented here.

The p_6 hazard RMSEs for both MRR and ML are presented in Figure 8. The RMSE increases with shape and decreases with sample size. The RMSE might be slightly less for ML estimates than MRR estimates. The results were broadly similar for the bias of the p_5, $p_{\bar{6}}$, and $p_{\bar{5}}$, hazard bias, so the results are not presented here. The relative magnitudes of the RMSE (Figure 8) and

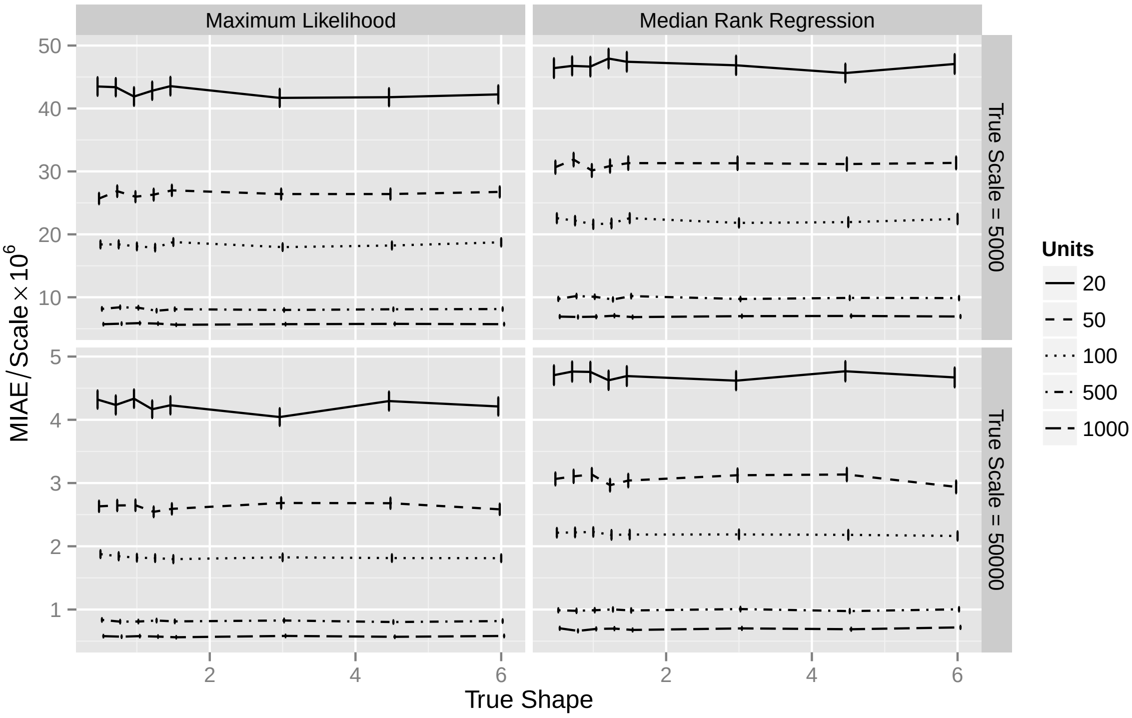

Figure 4: Mean integrated absolute error (MIAE) of the pdf, 100% fail. The y-axis has been scaled to facilitate comparison. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

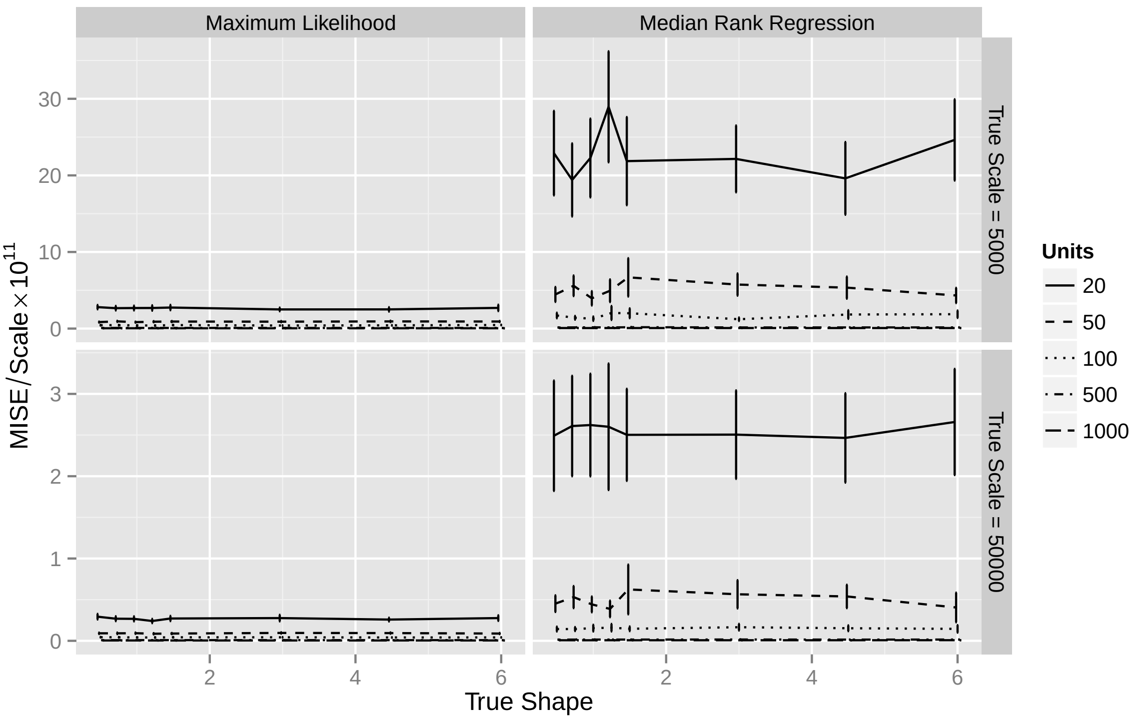

Figure 5: Mean integrated squared error (MISE) of the pdf, 100% fail. The y-axis has been scaled to facilitate comparison. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

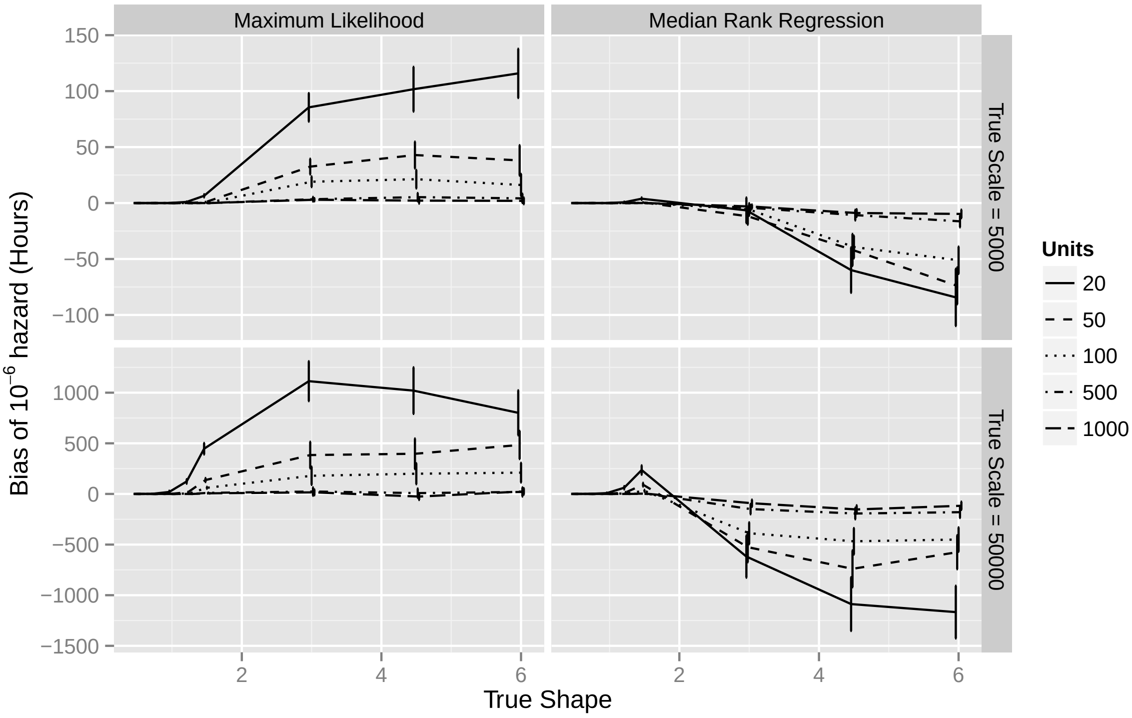

Figure 6: Bias of 10^{-6} hazard hour, 100% fail. Negative bias implies a conservative outcome: reaching the decision point earlier than is expected. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals. Missing points are too high to appear within the selected range.

bias (Figure 6) suggests that the standard deviation is larger than the bias for these simulations.

Finally, the p_6 risk ratios are presented in Figure 9. Here, the estimation algorithms show a substantial difference in behaviour. The ratios are more positive for the ML estimates than the MRR estimates, and expose the decision-maker to much higher risk, especially for low shape values. A potential cause of this greater risk can be see in Figure 2, which shows a consistent positive bias in the shape estimate for ML. The Weibull probability density shape shifts from monotonic to non-monotonic as the shape parameter increases through unity. The p_6 hazard cutoff value will always be 1 for sufficiently low values of shape (see Table 1), which are in any case operationally unrealistic in the current context, and positive bias will 'push' the mass of the distribution away from the zero point, hence increasing the hazard cutoff (see Figure 1). We would expect similar outcomes with increasing scale, for moderate to high values of shape. We also see an apparent change in direction of the risk ratio. We conjecture that this change is a result of the shift in the true shape parameter across the threshold of 1.

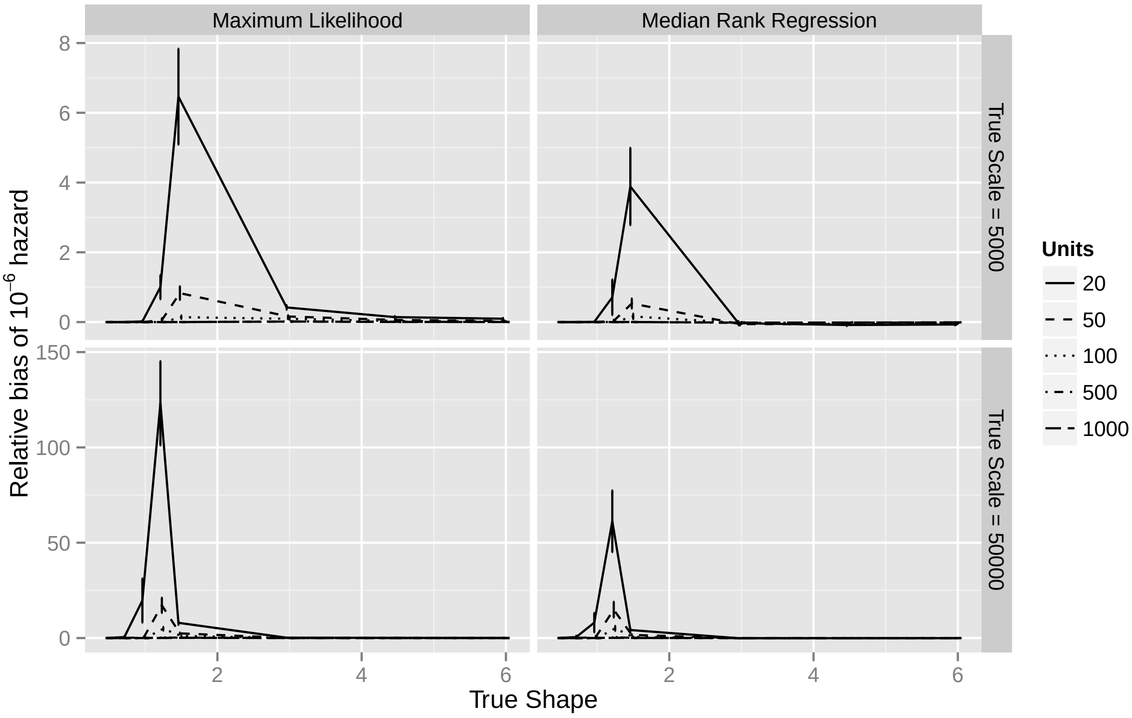

Figure 7: Relative bias of 10^{-6} hazard hour, 100% fail. Negative bias implies a conservative outcome: reaching the decision point earlier than is expected. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals. Missing points are too high to appear within the selected range.

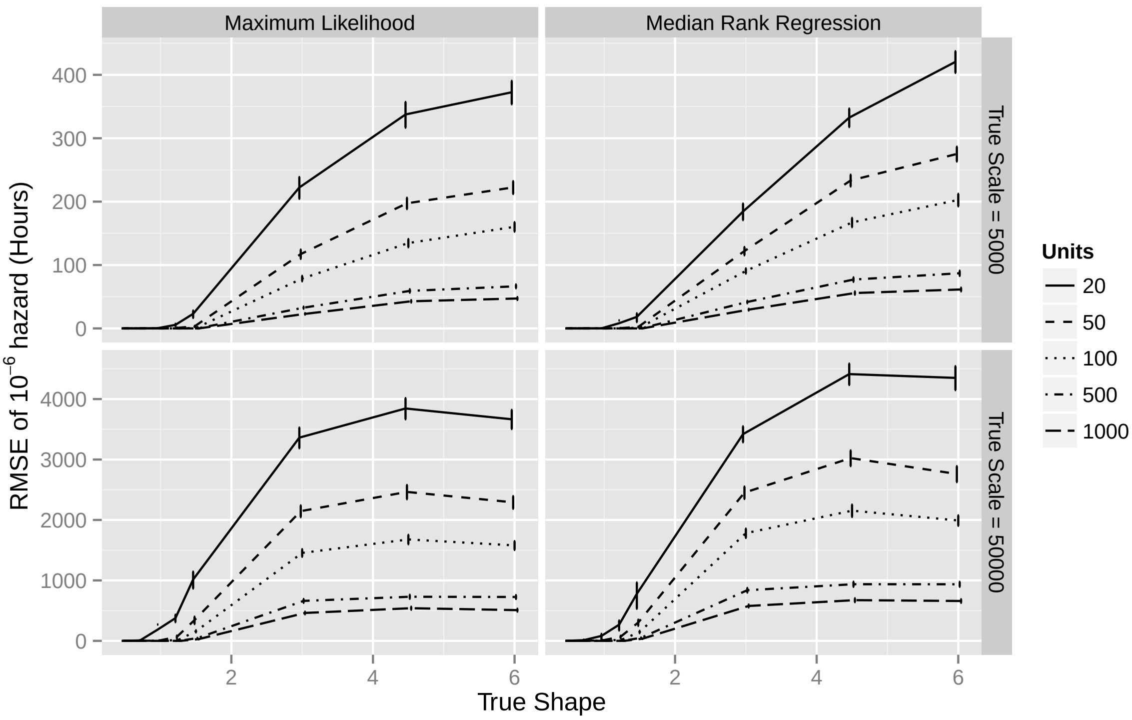

Figure 8: RMSE of 10^{-6} hazard hour, 100% fail. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

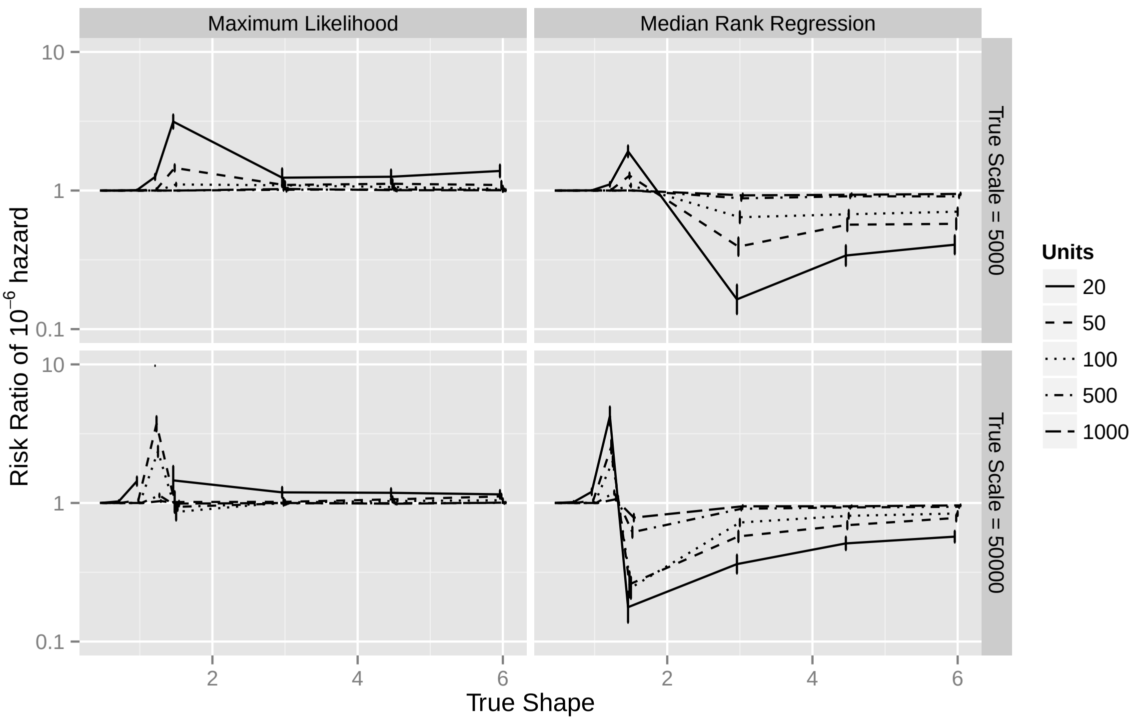

Figure 9: Geometric mean of 10^{-6} hazard hour risk ratio, 100% fail. Each point is the geometric mean of the ratio of the true cdf evaluated at the estimated hazard point and the actual hazard point for 1000 simulations; the vertical bars are 95% confidence intervals on the geometric mean. The y-axis is in logarithmic scale. Points that exceed the axes are omitted.

3.2 20% Fail

The purpose of the next reported set of simulations was to assess the impact of heavy censoring, specifically, 80%. The specifications used for this set of simulations are as follows.

- no staggering;
- four different scales (500, 2500, 5000, 50000);
- eight different shapes (0.5, 0.75, 1, 1.25, 1.5, 3, 4.5, 6);
- five different sample sizes (20, 50, 100, 500, 1000);
- 20% failures; and
- 1000 replicates.

Figure 10 provides the simulation results for the bias of the estimate of the shape parameter, α, for both ML and MRR. As before, the pattern is relatively constant across the range of scale, so results corresponding to only two values of scale are provided. The relative bias in estimates of α from both ML and MRR decreases towards 0 with increasing sample size, and is relatively constant across the range of true shape. The magnitude does not seem to change with true scale. The bias is positive for both ML and MRR, and seems to be about three times as large for the former as the latter.

It appears that the introduction of substantial censoring has substantially increased the bias of the MLE and had both increased and flipped the sign of the bias of the MRR estimates (c.f. Figure 2). However, this increase in bias may be due to the effective reduction in sample size that follows from censoring. For example, under 20% failures the effective sample size for 100 units is 20, and for 500 units is 100. Comparing the 500-unit and 100-unit lines in Figure 10 with the 100-unit and 20-unit lines respectively in Figure 2, the biases are similar, which suggests that the loss of information that arises from censoring is a substantial contributor to the extra bias.

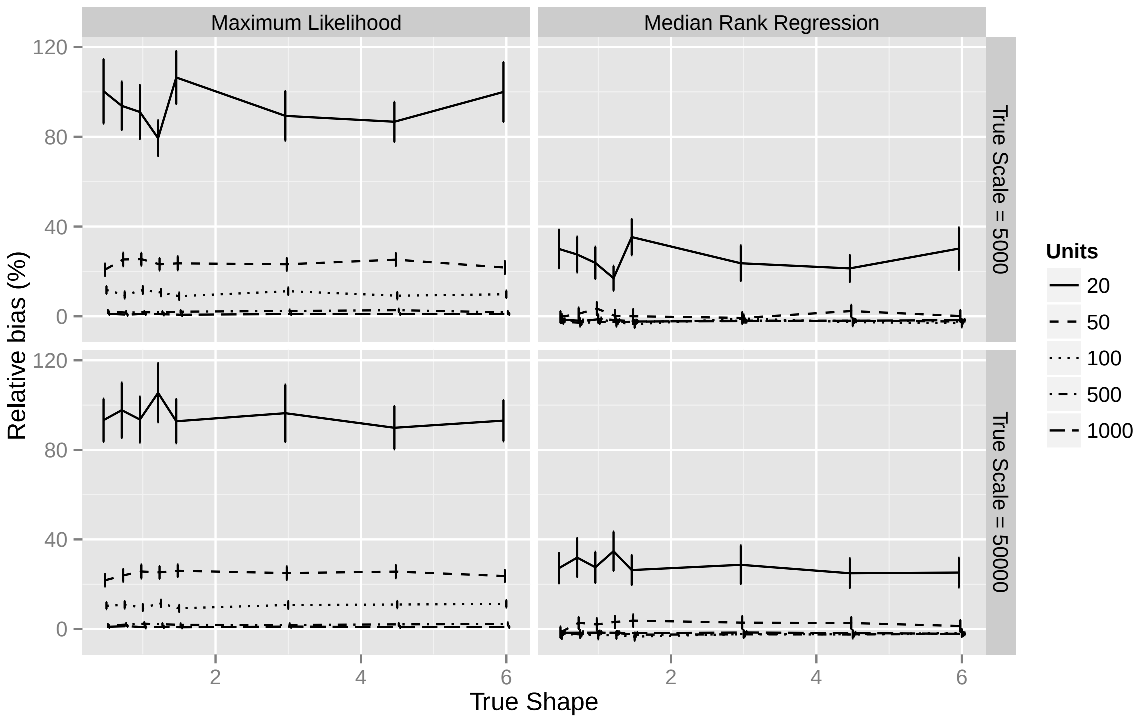

Figure 10: Relative bias of estimate of shape parameter α, 20% fail. Each point is the mean of $(\hat{\alpha} - \alpha)/\alpha \times 100$ from 1000 simulations; the vertical bars are 95% confidence intervals.

Figure 11 shows the simulation results for the bias of the estimate of the scale parameter, β, for both ML and MRR. The results are very similar to those in Figure 3. Again, the pattern is relatively constant across the range of scale, so results corresponding to only two values of scale are provided. Relative biases of scale for both ML and MRR are largely unaffected by true scale. Bias is positive for ML and reduces very close to zero with increasing sample size and increasing shape. Bias is positive but larger and decreases to close to zero with increasing sample size and shape for MRR.

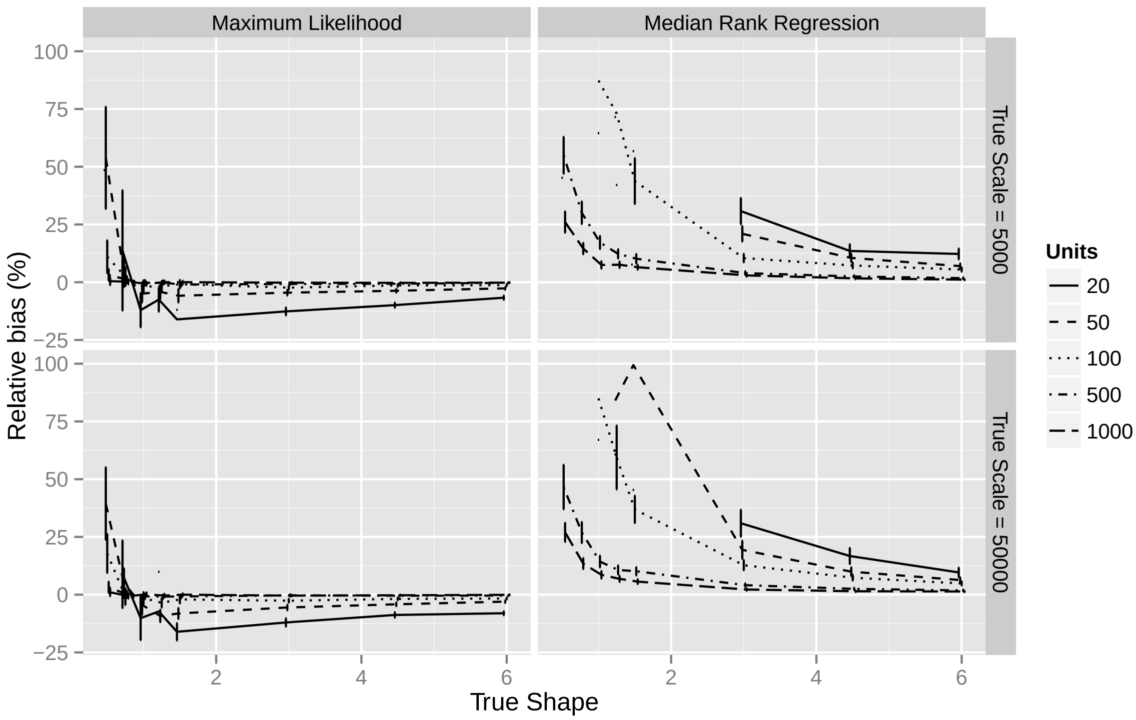

Figure 11: Relative bias of estimate of scale parameter β, 20% fail. Each point is the mean of $(\hat{\beta}-\beta)/\beta\times 100$ from 1000 simulations; the vertical bars are 95% confidence intervals. Points that exceed the axes are omitted.

Figures 12 and 13 provide the MIAE and MISE of the estimated distributions respectively. MIAE is independent of shape and decreases with true scale, reduces in magnitude and variability with increasing sample size, and is slightly higher for ML than MRR, which is different than observed in Figure 4. MISE shows similar patterns but is substantially lower for ML than MRR.

Figure 14 presents the p_6 hazard bias from the simulations. The biases arising from both estimates seem to increase with scale and decrease with sample size. The bias for ML is larger than MRR, and always positive, whereas the bias for the latter is often negative and comparably negligible. The change in bias from the uncensored case does not seem commensurate with the amount of censoring, that is, the bias for the 500-unit case in the 20% censored simulations is not close to the bias in the 100-unit case in the uncensored simulations, it is instead substantially larger (c.f. Figure 6). As before, the results were broadly similar for the bias of the p_5, $p_{\bar{6}}$, and $p_{\bar{5}}$, hazard bias, so the results are not presented here.

The p_6 hazard RMSEs for both MRR and ML are presented in Figure 16. As before, the RMSE increases with shape and decreases with sample size. The RMSE now might be slightly less for MRR estimates than ML estimates, especially in the range of 1–5 for shape. The results were broadly similar for the bias of the p_5, $p_{\bar{6}}$, and $p_{\bar{5}}$, hazard bias, so the results are not presented here. The relative magnitudes of the RMSE (Figure 16) and bias (Figure 14) suggests that the standard deviation is larger than the bias for these simulations. A visual comparison of Figures 8 and 16 suggests that the censoring has approximately doubled the RMSE of the

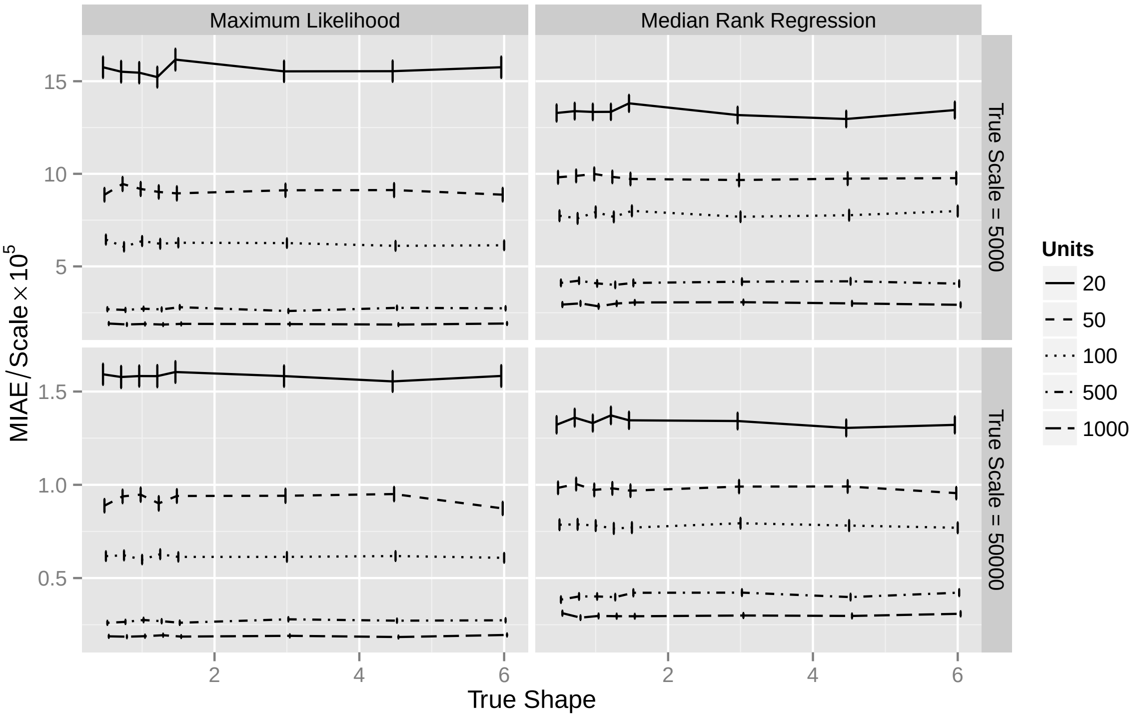

Figure 12: Mean integrated absolute error (MIAE) of the pdf, 20% fail. The y-axis has been scaled to facilitate comparison. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

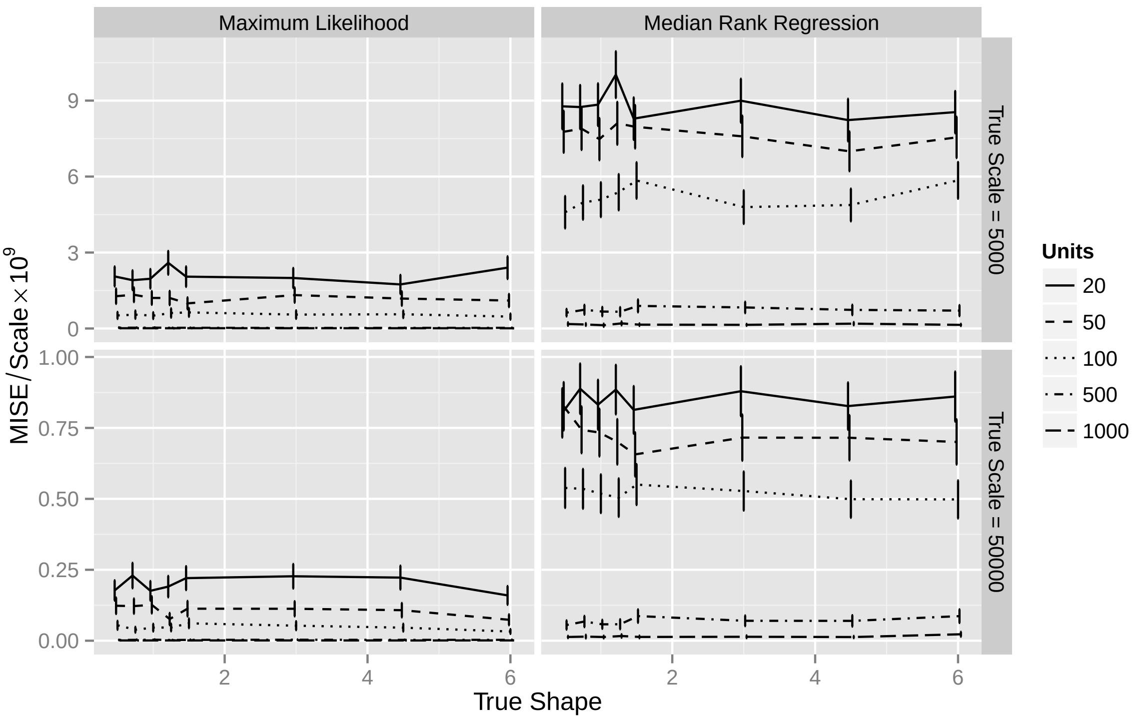

Figure 13: Mean integrated squared error (MISE) of the pdf, 20% fail. The y-axis has been scaled to facilitate comparison. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

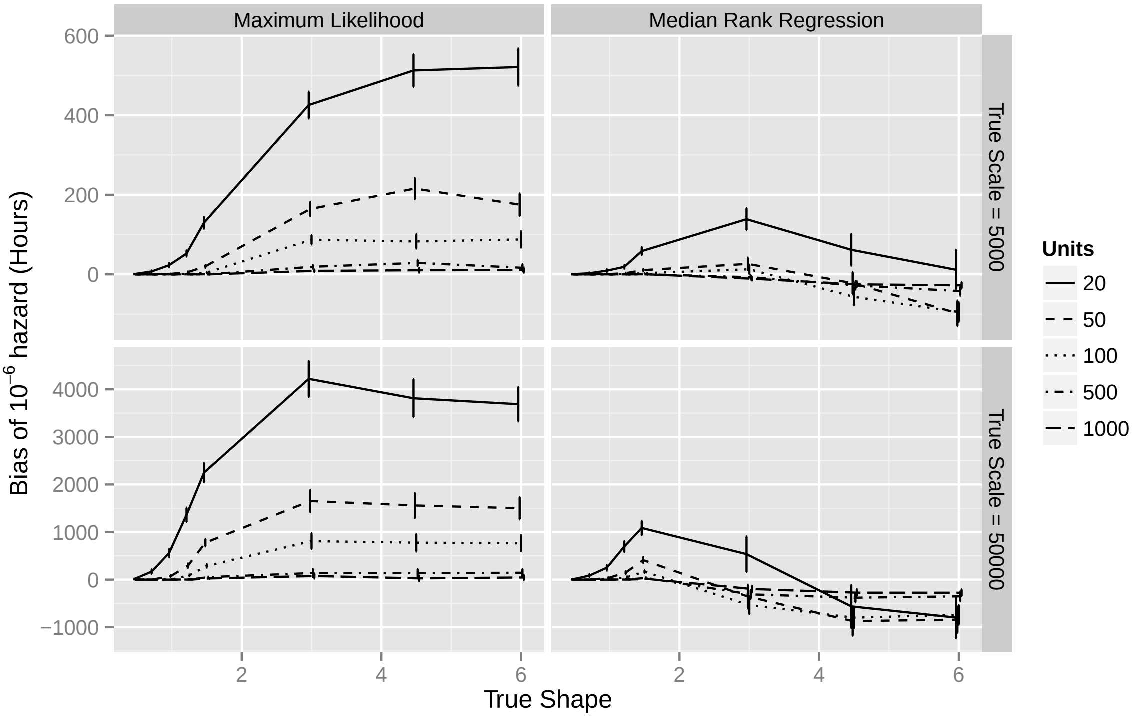

Figure 14: Bias of 10^{-6} hazard hour, 20% fail. Negative bias implies a conservative outcome: reaching the decision point earlier than is expected. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

hazard cutoff.

Figure 15 presents the p_6 hazard *relative* bias from the simulations. The relative biases arising from both estimates seem to increase with scale, decrease with sample size, and increase initially and then descend with shape. The location of the peak relative to true shape seems to change with true scale. The relative bias seems larger in absolute value for ML estimates than MRR estimates. The results were broadly similar for the bias of the p_5, $p_{\bar{6}}$, and $p_{\bar{5}}$, hazard bias, so the results are not presented here.

The p_6 risk ratios are presented in Figure 17. Again, the estimation algorithms show a substantial difference in behaviour. The ratios are more positive for the ML estimates than the MRR estimates, and expose the decision-maker to much higher risk, especially for low shape values. As before, a potential cause of this greater risk can be see in Figure 10, which shows a consistent positive bias in the shape estimate for ML, see p. 18 for discussion. As before, we see an apparent change in direction of the risk ratio. We again conjecture that this change is a result of the shift in the true shape parameter across the threshold of 1. Comparing Figures 9 and 17 suggests that the 80% censoring greatly amplifies the risk.

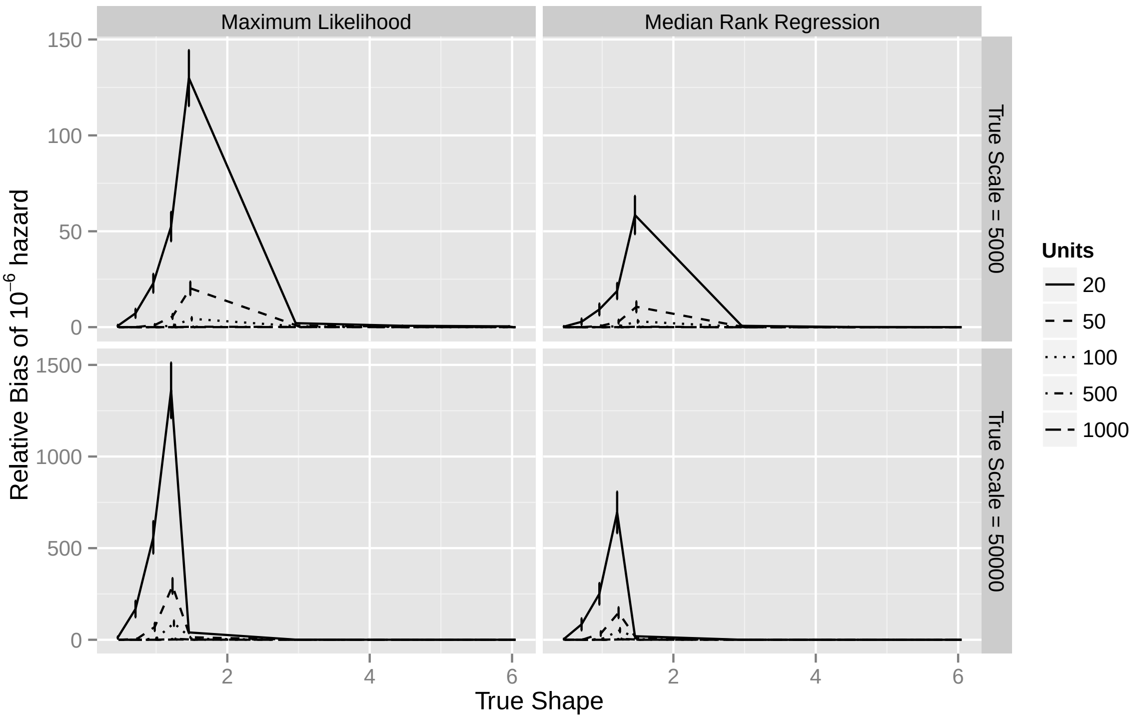

Figure 15: Relative bias of 10^{-6} hazard hour, 20% fail. Negative bias implies a conservative outcome: reaching the decision point earlier than is expected. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

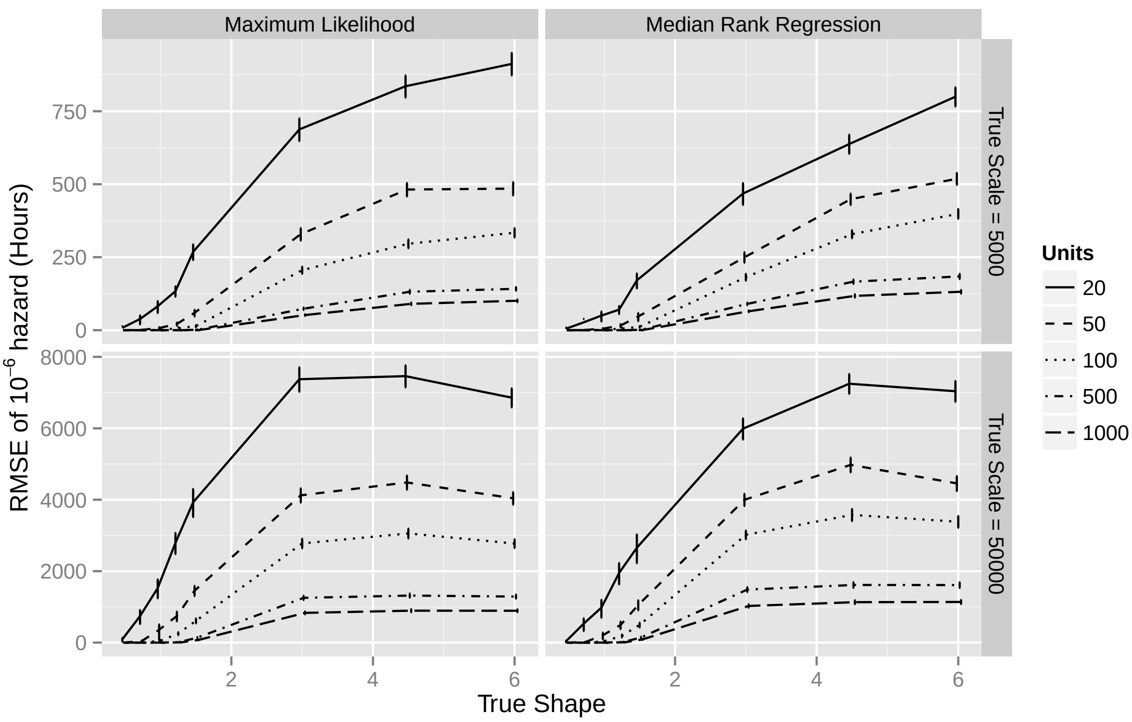

Figure 16: RMSE of 10^{-6} hazard hour, 20% fail. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

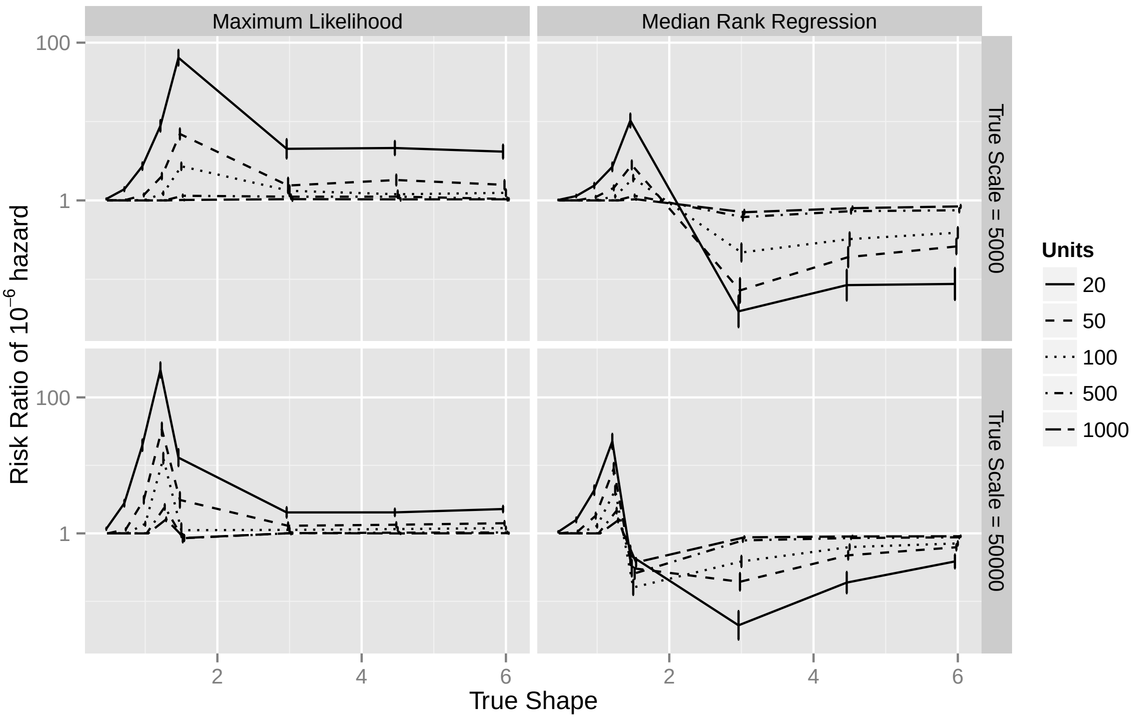

Figure 17: Geometric mean of 10^{-6} hazard hour risk ratio, 20% fail. Each point is the geometric mean of the ratio of the true cdf evaluated at the estimated hazard point and the actual hazard point for 1000 simulations; the vertical bars are 95% confidence intervals on the geometric mean. The y-axis is in logarithmic scale.

3.3 Bespoke Scenario 1 — Lowball

The bespoke 'lowball' (low number of units) simulation is the first intended to provide insights into scenarios that are operationally realistic. In terms of the simulation, the differences between this and the previous simulations are not important, however, the differences in presentation are substantial. Here we have only one sample size, so we collapse linked metrics into frames within a plot (previously the frames reported different values of scale), and report different scales using different colours inside the frame (previously we used colours inside frames to distinguish between the sample sizes). Therefore the patterns that we have observed in the previous simulations will not necessarily be reproduced here. The specification for the reported simulation follows.

- Failing units replaced by units with same distribution of lifetime;
- four different scales (500, 2500, 5000, 50000);
- eight different shapes (0.5, 0.75, 1, 1.25, 1.5, 3, 4.5, 6);
- 50 units;
- 6% (stop at 3-*rd* fail); and
- 1000 replicates.

It is worth emphasizing the point that for each of these simulations, the estimates of the Weibull parameters are being made after the third fail from 50 units, which is a situation in which very little information is available about the process parameters.

Figure 18 presents the relative bias for the estimators of both of the Weibull parameters; shape in the upper row and scale in the lower. The y-axis had to be truncated in order to allow a reasonable comparison; values exceeding the axis limits are not plotted.

The relative bias in shape estimates is high for both MRR and ML, but much more so for the latter — more than twice as high. The relative bias of shape is independent of the true shape and scale, which agrees with our earlier results (Figures 2 and 10).

The relative bias in scale estimates is high for both MRR and ML, but much more so for the former. The relative bias of scale reduces with increasing shape, which again agrees with our earlier results (Figures 3 and 11). There is no evidence of an effect of true scale upon the relative bias in scale estimates.

Figure 19 presents the MIAE and MISE for the distributions arising from the parameter estimates. The metrics disagree on which estimator is preferred; MIAE prefers MRR, and MISE prefers ML. The quadratic loss (L_2) function embedded in MISE will penalize remote outlying areas more heavily than will the absolute (L_1) loss for MIAE. These results suggest that the ML-inspired estimated distribution does not have such extreme deviations as does the MRR estimated distribution, but for the most part, the MRR distribution is closer.

The risk ratio and the RMSE for the hazard cutoff are both presented in Figure 20. The results show that for low shape, the risk ratios for ML and MRR are both higher than 1, with ML being higher than MRR. At high shape values the risk for ML remains greater than 1, whereas the risk for MR is less than 1, and approximately proportionately so. The RMSE patterns for the two estimators are similar, with MRR estimates having slightly smaller RMSE than the ML estimates. As before, we see an apparent change in direction of the risk ratio. We again conjecture that this change is a result of the shift in the true shape parameter across the threshold of 1.

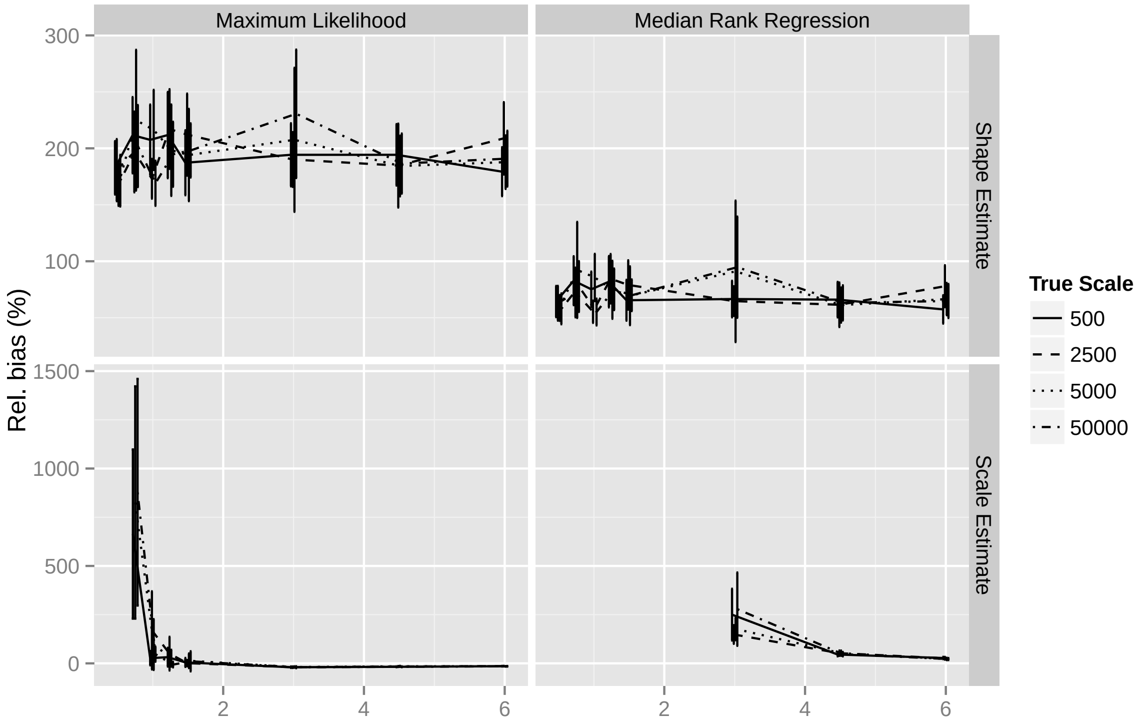

Figure 18: Relative bias of shape and scale parameter estimates; 50 units, 6% fail, failing units are replaced. The top row reports shape (α) and the bottom row, scale (β). Biases greater than 1000% are ignored. The vertical bars are 95% confidence intervals. Points that exceed the axes are omitted.

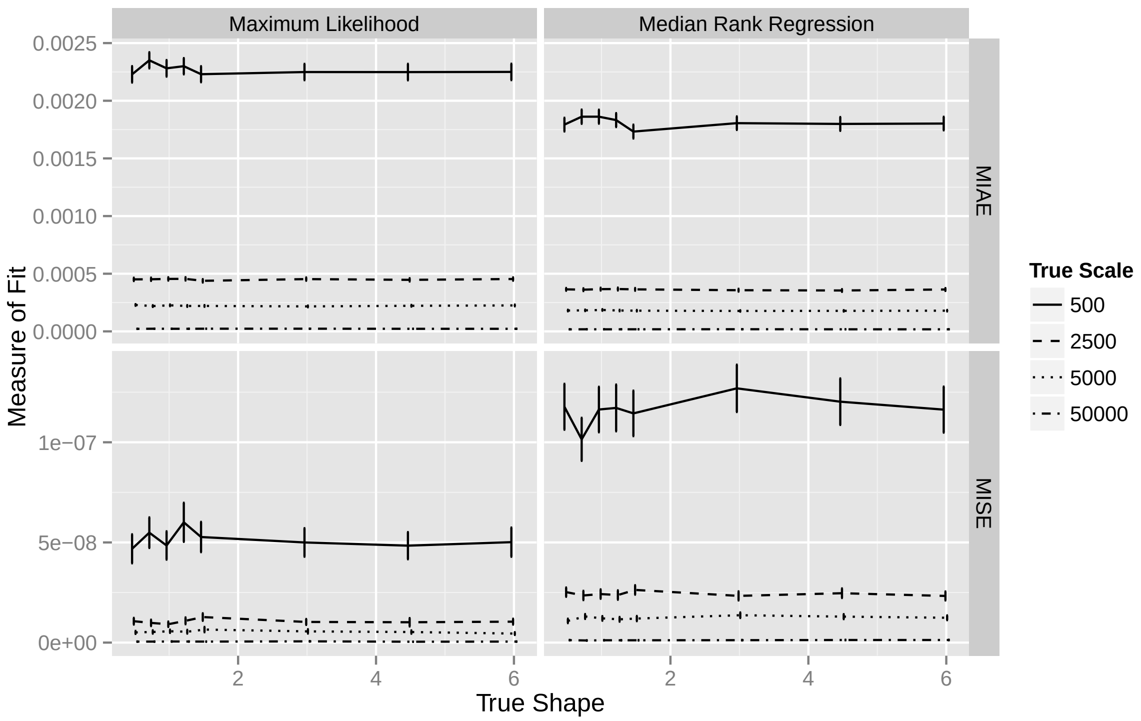

Figure 19: MIAE and MISE of the pdf; 50 units, 6% fail, failing units are replaced. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

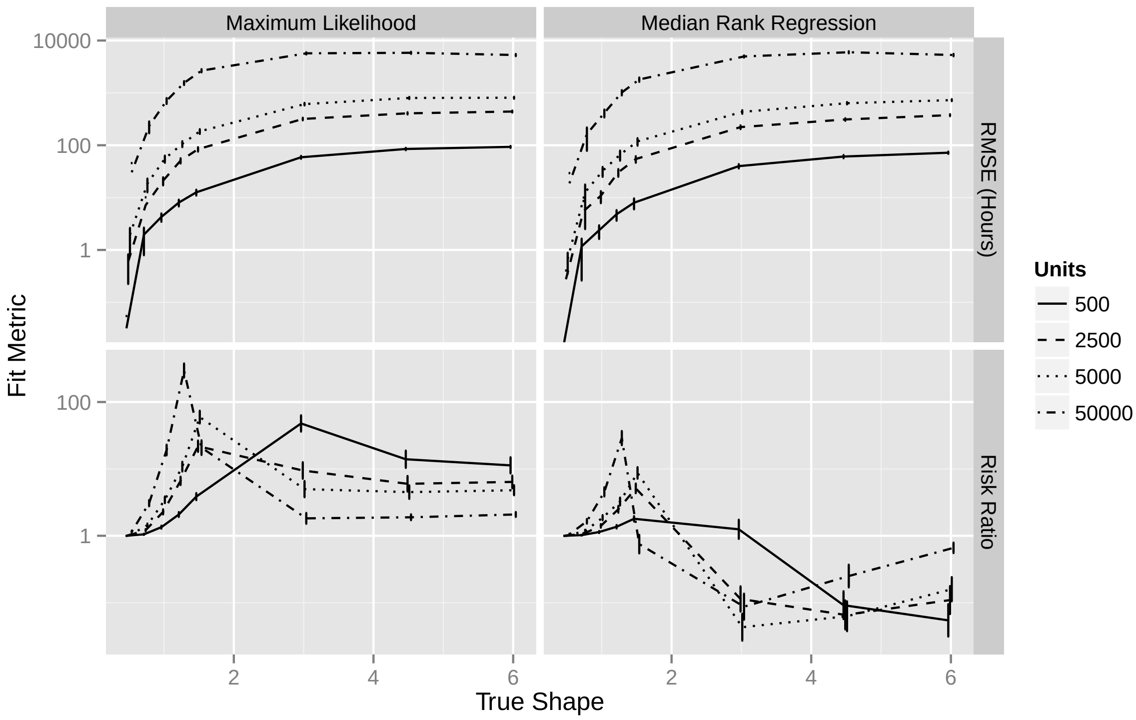

Figure 20: 6% fail, failing units are replaced. The first plot reports the RMSE of 10^{-6} hazard hour. The second plot presents the geometric mean of 10^{-6} hazard hour risk ratio. The vertical bars are 95% confidence intervals on the geometric mean. The y-axis is in logarithmic scale.

3.4 Bespoke Scenario 2 — Highball

The bespoke 'highball' (high number of units) simulation is the second intended to provide insights into scenarios that are operationally realistic. We present similar graphical diagnostics as in the previous section. The specification for the reported simulation follows.

- Failing units replaced by units with same distribution of lifetime;
- four different scales (500, 2500, 5000, 50000);
- eight different shapes (0.5, 0.75, 1, 1.25, 1.5, 3, 4.5, 6);
- 500 units;
- 1% (stop at 5-*th* fail); and
- 1000 replicates.

We emphasize that for each of these simulations, the estimates of the Weibull parameters are being made after the fifth fail from 500 units, which is a situation in which very little information is available about the process parameters.

Figure 21 presents the relative bias for the estimators of both of the Weibull parameters; shape in the upper row and scale in the lower. The y-axis had to be truncated in order to allow a reasonable comparison; values exceeding the axis limits are not plotted.

The relative bias in shape estimates is high for both MRR and ML, but much more so for the latter — more than four times as high. The relative bias of shape is independent of the true shape and scale, which agrees with our earlier results (Figures 2 and 10).

The relative bias in scale estimates is high for both MRR and ML, but much more so for the former. The relative bias of scale reduces with increasing shape, which again agrees with our earlier results (Figures 3 and 11). There is no evidence of an effect of true scale upon the relative bias in scale estimates.

Figure 22 presents the MIAE and MISE for the distributions arising from the parameter estimates. As before, the metrics disagree on which estimator is preferred; MIAE prefers MRR, and MISE prefers ML.

The risk ratio and the RMSE for the hazard cutoff are both presented in Figure 23. The results again show that for low shape, the risk ratios for ML and MRR are both higher than 1, with ML being higher than MRR. At high shape values the risk for ML remains greater than 1, whereas the risk for MR is less than 1, and approximately proportionately so, except for the highest scale (50000), which resulted in a slightly positive risk ratio for MRR. The RMSE patterns for the two estimators are similar, with MRR estimates having slightly smaller RMSE than the ML estimates, except for the highest scale (50000). As before, we see an apparent change in direction of the risk ratio. We again conjecture that this change is a result of the shift in the true shape parameter across the threshold of 1.

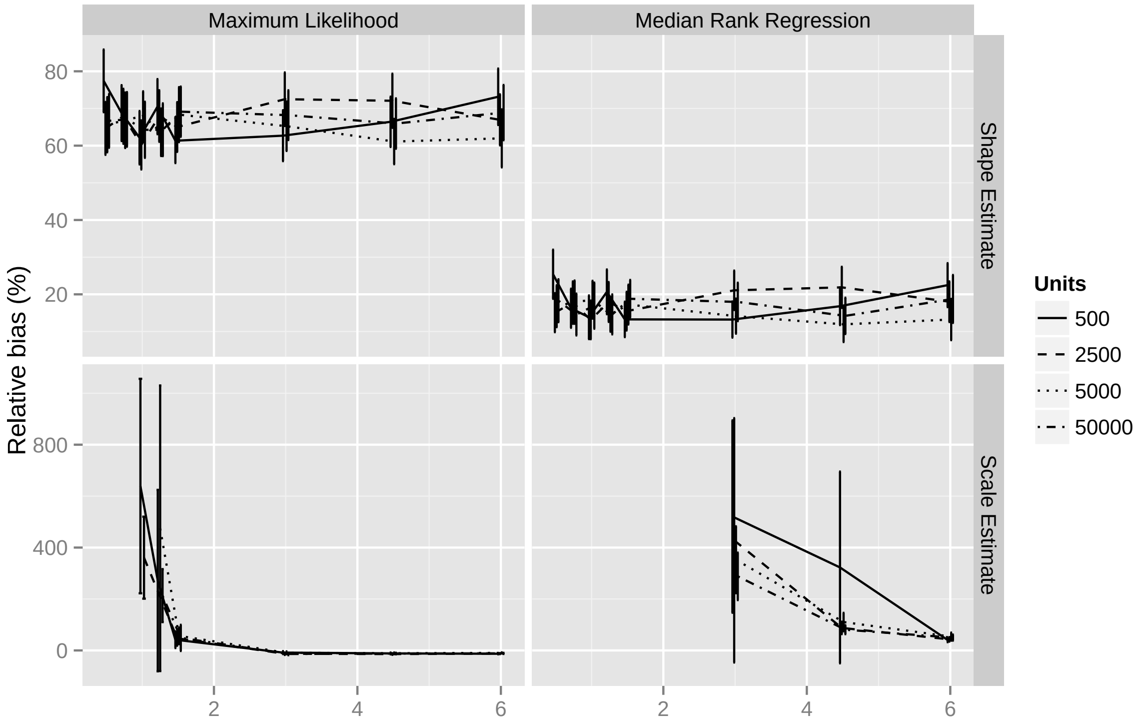

Figure 21: Relative bias of shape and scale parameter estimates; 500 units, 1% fail, failing units are replaced. The top row reports shape (α) and the bottom row, scale (β). Biases greater than 1000% are ignored. The vertical bars are 95% confidence intervals. Points that exceed the axes are omitted.

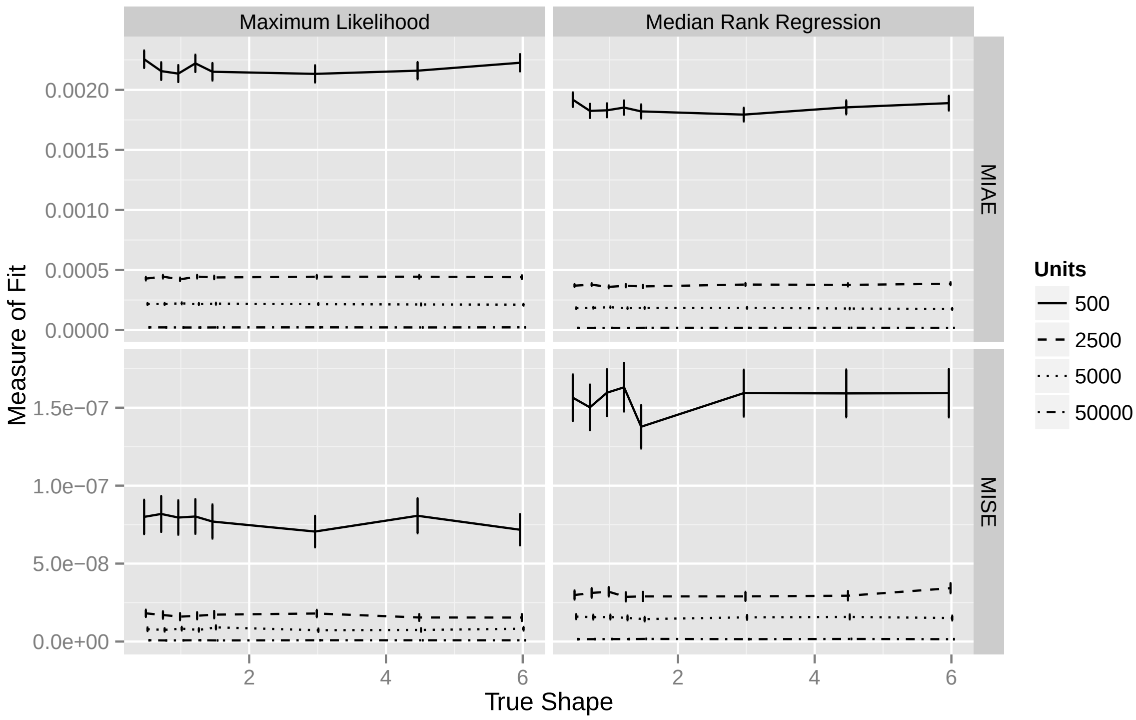

Figure 22: MIAE and MISE of the pdf; 500 units, 1% fail, failing units are replaced. Each y-axis has been scaled to facilitate comparison. Each point is the mean of 1000 simulations; the vertical bars are 95% confidence intervals.

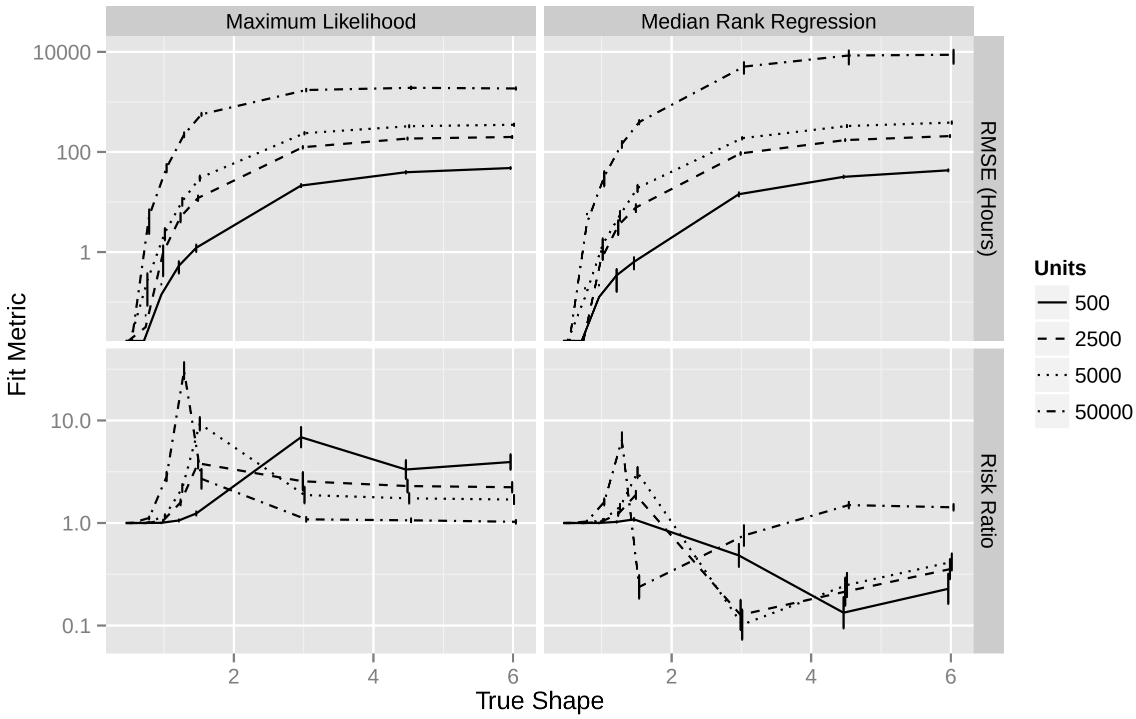

Figure 23: 500 units, 1% fail, failing units are replaced. The first plot reports the RMSE of 10^{-6} hazard hour. The second plot presents the geometric mean of 10^{-6} hazard hour risk ratio. The vertical bars are 95% confidence intervals on the geometric mean. The y-axis is in logarithmic scale.

4 Discussion

Genschel and Meeker (2010) provide an overview of a number of studies (their Table 1), but as they observe, the variation in the combinations of (i) fit quality statistics, (ii) censoring types and depths, (iii) staggering arrangements, and (iv) sample sizes make a direct comparison problematic. Where our results are directly comparable with previous studies, they are largely in agreement, with one exception. Olteanu and Freeman (2010) found, as we did, that the bias for α was lower under MRR than ML, and that the ML bias was positive; and furthermore the bias for estimating β was lower under ML than MRR. However, their MRR bias for β was negative.

The results of our simulations are discussed below. Here we focus on the metrics for P_6 only; the results were broadly similar for P_5, $P_{\bar{6}}$, and $P_{\bar{5}}$, so we do not discuss them further.

In the uncensored case, the P_6 hazard bias and RMSE for both estimates increased with true scale, decreased with sample size, and increased with true shape (Figures 6, 8, and 9). These results seem intuitively reasonable: as scale increases, the distribution stretches along the x-axis so the corresponding errors in identifying a location on the axis will dilate. As the sample size increases, the fidelity of the fitted distribution to the true distribution will increase. Furthermore, as the shape of the distribution increases it switches from being monotonic decreasing and becomes more symmetric. When the distribution is monotonic decreasing, the hazard cutoff is very early, usually by definition directly after the first hour, which is not an operationally interesting result. An alteration in the shape of the distribution towards symmetry will lengthen the left tail, which will have a comparable effect upon the magnitude of error to increasing the scale.

The hazard bias was slightly larger in absolute value for ML than MRR, and positive for ML but negative for MRR, and the hazard RMSE was slightly less for ML than MRR. Accordingly, the risk ratios were negative for MRR and positive for ML. If quadratic closeness to the correct cutoff hour is important then these results support ML over MRR. However, positive bias in estimating the cutoff will lead to maintenance being delayed relative to policy, which exposes the fleet to greater risk than intended, whereas negative bias leads to maintenance being carried out too early, at an arguably different loss. A detailed exploration of the asymmetry of loss in this instance is beyond the scope of this paper, so we merely remark that it is possible that a negatively biased, less precise estimator might be preferred over a positively biased but more precise estimator.

The results differ sharply under heavy censoring. The positive bias of the ML cutoff estimate increases greatly, and that of MRR becomes attenuated (Figure 14, compare with Figure 6). The RMSE of the estimates are more similar, except in the lowest sample size, for which MRR outperforms ML (Figure 16, compare with Figure 8). Finally, the risk ratio for MRR under heavy censoring is uniformly less than that for ML (Figure 17, compare with Figure 9). Collectively these results suggest that MRR performs better under heavy censoring: the bias, RMSE, and risk ratio all decrease relative to ML.

We now try to provide the context for this observation. It appears that the introduction of substantial censoring substantially increased the bias of the MLE and both increased and flipped the sign of the bias of the MRR estimates (c.f. Figure 2). This increase in bias may simply be due to the effective reduction in sample size that results from censoring. For example, under 20% failures the effective sample size for 100 units is 20, and for 500 units is 100. Comparing the 500-unit and 100-unit lines in Figure 10 with the 100-unit and 20-unit lines respectively in Figure 2, the biases are similar, which suggests that the loss of information that arises from censoring is a substantial contributor to the extra bias.

In terms of estimation of percentiles, which is similar to but not the same as the identification of hazard cutoffs, Olteanu and Freeman (2010) found that "MRR tends to underpredict for the large number of failures and MLE (sic) tends to over-predict for nearly every case considered", and that the relative performance of the estimators was a function of true shape. These results are consonant with our observations regarding the predictions of the hazard cutoff.

We now summarize results that shed some light on the comparison of MRR and ML for estimating the hazard cutoff. The hazard cutoff in this instance is invariably located deep in the left tail. Intuitively, the scale of the distribution has less impact than does the shape on the behaviour of the left tail, so we would expect accurate estimates of the shape to be more important. As previous authors found, the relative bias of shape for both ML and MRR was unaffected by true shape or scale (Figures 2 and 10). Bias was positive for ML and reduces very close to zero with increasing sample size, and is negative and smaller in absolute value and increases to quite close to zero within increasing sample size for MRR (Figure 2). Also, the relative bias of scale for both ML and MRR was unaffected by true scale, and decreases with true shape (Figures 3 and 11). Bias was positive for ML and reduced very close to zero with increasing sample size. Bias was positive but larger and decreased to close to zero with increasing sample size and shape for MRR. These results about the shape and scale estimates partially help to explain the hazard cutoff results of the different algorithms. MRR estimates of shape have better performance than ML estimates of shape, and we would expect that better performance to translate to functions of the parameters that are more influenced by the shape than the scale.

4.1 Future Directions

We now briefly summarize potential future directions. The key point moving ahead is that if the purpose of estimating the Weibull parameters is in order to use them to make a decision from a computed hazard cutoff, then the estimation approach should focus on the estimation of that cutoff. That is, the end use of the parameter estimates should be kept in focus.

4.1.1 Quantifying uncertainty

As Genschel and Meeker (2010) point out, point estimates by themselves have limited utility, although they and most of the other simulation studies reviewed do focus on the statistical properties of the point estimates. A further layer of risk-based decision-making could be implemented if interval estimates of the decision-supporting functions of the parameters were available. We did not assess any statistical properties of interval estimates, mainly because there is no formal way in which such information can be integrated into the existing decision process, but they are a potentially important topic.

4.1.2 Other parameter estimation approaches

The work reported here shows the importance of considering the use to which estimates will be put in comparing the utility of estimators. The decision that underpins the estimation of the Weibull parameters is one that requires accurate estimation of the behaviour of the left tail of the distribution. The left tail of the distribution is highly sensitive to shape (see figure 1), and we conjecture that the improved performance of MRR relative to ML derives from its better estimation of the shape parameter (see., e.g., figures 2 and 10). Future work should try to verify and then focus on the behaviour of the left tail, including possibly the use of a base distribution that has a narrower domain of underlying shapes, or a non-parametric approach. For example, Quigley and Revie (2011) focus on estimating probability of rare events when no failures have been observed.

In situations where prior information is known about failures, and especially in situations with a small number of failure events, a Bayesian approach may provide better estimates (Heitjan et al., 2004; Watson et al., 2004; Olteanu and Freeman, 2010; Ahmed et al., 2012; Danish and Aslam, 2014; Guo et al., 2014). Wu et al. (2014) and Guure et al. (2013) both found that Bayesian estimates perform better than MLEs in terms of mean-square errors (MSE) and absolute bias, using Type-1 progressive hybrid censoring and interval censoring respectively. In a study to assess in-orbit failures of small satellites, a novel method using Bayesian theory and Markov Chain Monte Carlo simulations provided Weibull models that were better fitted than ML, evaluated using coefficients of determination, and was proposed as a valuable method where there is a large amount of censored data with small numbers of failures (Guo et al., 2014). Erto (1982) and Erto et al. (2010) provide a Bayesian algorithm and an application to wind data respectively. Nelson (1985) provides an early, not-quite-Bayesian algorithm for estimation with few or no failures and prior information on the shape parameter α.

Alternatively, numerous studies have led to suggestions for bias corrections of the estimates of Weibull parameters. Such corrections may affect the behaviour of the estimators in promising ways. For example, Montanari et al. (1997) used a simulation study to show that MRR and ML may give rise to significant bias errors, and recommended modifications to the estimators. Hirose (1999) suggests bias correction methods for ML, see also Cacciari et al. (1996). van Zyl and Schall (2012) suggest weights for MRR; see also Faucher and Tyson (1988). Ageel (2002) proposed a new method for estimating Weibull quantiles that was superior in MSE except for the lower quantiles, which reduces its likely utility in the present case. Yang and Scott (2013b) provide robust estimates of the Weibull parameters that may result in improved performance; R code is available (Yang and Scott, 2013a).

Furthermore, the well-established history of bespoke corrections or tweaks to likelihood estimates suggests that important improvement may be possible for any given situation, including both small-sample asymptotics (see, e.g., Pawitan, 2001; Brazzale et al., 2007; Butler, 2007) and residual maximum likelihood (ReML, see, e.g., Laird and Ware, 1982; Robinson, 1991; Pinheiro and Bates, 2000).

5 Conclusions

5.1 Outcomes

1. MRR performed better than ML for parameter estimation for the Weibull distribution using uncensored and heavily right, type-2 censored data, where the basis for comparison was the realized ratio of risk, described in Section 2.1.1.

5.2 Recommendations

1. MRR should be used rather than ML if the purpose of estimation is to guide the decision of when to withdraw a fleet for maintenance.
2. Further work should be undertaken to develop point and interval estimators with improved performance in the left tail.

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A Appendix: R scripts

A.1 1_Functions.R

```
# install required packages

library(MASS)
library(survival)
library(lattice)

mail.and.stop <- function() {
  mail("mensurationist@gmail.com",
       "Sims on server: ERROR.",
       "Stopped.")
  stop()
}

mail.and.browse <- function() {
  mail("mensurationist@gmail.com",
       "Sims on server: ERROR.",
       "Stopped.")
  recover()
}

options(error = mail.and.browse)

options(error = mail.and.stop)

### FUNCTION TO SIMULATE FAILED/CENSORED DATA BETWEEN LINES 20 - 100
### EXAMPLE OF SIMDAT FUNCTION BETWEEN LINES 110 - 119

### MLE FUNCTION BETWEEN LINES 130 - 155
### EXAMPLE OF MLE FUNCTION BETWEEN LINES 160 - 165

### MRR FUNCTION BETWEEN LINES 180 - 245
### EXAMPLE OF MRR FUNCTION BETWEEN LINES 250 - 254 (COMMENTED OUT)

# define weibull.staggered function (modified from code provided by Spoon)
# used to simulated times of failed and censored data
################################################################################
#
# Function to generate staggered and non-staggered Weibull random numbers
# using Nick's oddball definition of staggering, not Andrew's nice one.
#
# Authors: JP, NA, SW, AR.
#
# 30/04/2012
#
################################################################################

# define MLE function
```

```
mle.func <- function(time,
                     censored,
                     p.start = c(3, 50),
                     check = FALSE,
                     LL = FALSE
                     ) {
# use survreg function in library(survival) package
# to fit MLE of sampleData and sampleStatus
#  fitDist <- survreg(formula = Surv(time, 1 - censored) ~ 1,
#                      dist = "weibull")
# Parameterisation of Weibull distribution is different between
# rweibull and the survival package
#  surv.shape.est <- 1 / fitDist$scale
#  surv.scale.est <- exp(coef(fitDist))
  nloglik <- function(p) {
    nll <- -sum(ifelse(censored,  ## Using log link functions.
                       suppressWarnings(pweibull(time, exp(p[1]), exp(p[2]),
                                                 lower.tail = FALSE,
                                                 log.p = TRUE)),
                       suppressWarnings(dweibull(time, exp(p[1]), exp(p[2]),
                                                 log = TRUE))))
    if(is.finite(nll)) nll else .Machine$double.xmax
  }
  p.start <- log(p.start)
#  print(p.start)
#  cat("\n")
#  spoon <- nlm(nloglik, p = p.start, gradtol = 1e-12)
  me1 <- optim(p.start,
               nloglik,
               control = list(
                   maxit = 10000,
#                   parscale = p.start,
                   reltol = .Machine$double.eps))
  me2 <- optim(p.start,
               nloglik,
               method = "BFGS",
               control = list(
                   maxit = 10000,
#                   parscale = p.start,
                   reltol = .Machine$double.eps))
#  me3 <- nlminb(start = p.start,
#                nloglik,
#                control = list(
#                    iter.max = 10000,
#                    eval.max = 10000,
#                    rel.tol = 1e-12))
#  cat(spoon$code, "\t", me1$convergence, "\t", me2$convergence, "\t",
#      me3$convergence, "\n")
#  results <- c(
#    surv.shape.est, surv.scale.est, fitDist$loglik[1],
#    spoon$estimate, -spoon$minimum,
```

```
#    me1$par, -me1$value,
#    me2$par, -me2$value,
#    me3$par, -me3$objective)
#  if (check)
#    if (var(results[c(1,4,7,10)]) > 0.5 |
#        var(results[c(2,5,8,11)]) > 0.5) browser()
  results <-
    list(#c(surv.shape.est, surv.scale.est),
#         exp(spoon$estimate),
         exp(me1$par),
         exp(me2$par))#,
#         exp(me3$par))
  if (LL) {
    return(c(results[[which.min(c(#fitDist$loglik[1],
#                                  -spoon$minimum,
                                 -me1$value,
                                 -me2$value))]],#,
#                                  -me3$objective))]],
             min(c(#fitDist$loglik[1],
#                   -spoon$minimum,
                  -me1$value,
                  -me2$value))))#,
#                   -me3$objective))))
  } else {
      return(c(results[[which.min(c(#fitDist$loglik[1],
#                                    -spoon$minimum,
                                   -me1$value,
                                  -me2$value))]]))#,
#                                  -me3$objective))]]))
  }
} # end mle.func





weibull.staggered <-
  function(numComponents, # numComponents = vector of sample size
           shape,   # shape = vector of shape parameter of the Weibull, beta
           scale,   # scale = scale parameter of the Weibull, eta (hours)
           propFailed,    # propFailed = number of sample that fails
           niter = 10,     # number of times components can be replaced
           stagger = FALSE,   # Whether or not to stagger the censored data
           pause = FALSE) {   # Whether or not to stop to look around
  ## Simulate time to failure and censoring
  tm <- tout <- rweibull(numComponents, shape = shape, scale = scale)
  ## Check whether staggered or not
  if (stagger > 1) { # If staggered
    ## Find the census time - always at least two fails
    Tmax <- sort(tm)[max(2, ceiling(propFailed*numComponents))]
    ## Set censoring flag
```

```
    censored <- tm > Tmax
    r <- rank(tm, ties = "first")
    ## Find the quantile corresponding to the censoring point
    Wmax <- pweibull(Tmax, shape = shape, scale = scale)
    ## Generate random censoring point, higher than Wmax
    u.v <- runif(sum(censored)) * runif(1, Wmax, 1)
    # WAS:
    # u.v <- runif(sum(censored)) * Wmax
    ## Feed the uniforms into the Weibull inverse cdf
    tm[censored] <- qweibull(u.v, shape = shape, scale = scale)
    ## Rerank
    r <- rank(tm, ties = "first")
  } else if (stagger == 1) { # Component replacement staggering
    replaced <-
      matrix(rweibull(numComponents * niter,
                      shape = shape, scale = scale),
             ncol = niter)
    times <- cbind(tm, replaced)
    history <- t(apply(times, 1, cumsum))
    Tmax <- sort(history)[max(2, ceiling(propFailed * numComponents))]
    while(min(apply(history, 1, max)) <= Tmax) {
      times <- cbind(times, matrix(rweibull(numComponents * niter,
                                            shape = shape, scale = scale),
                                   ncol = niter))
      history <- t(apply(times, 1, cumsum))
    }
    is.na(history[history > Tmax]) <- TRUE
    fail.obs <- times[!is.na(history)]
    censored.obs <- Tmax -
      apply(history, 1,
            function(x) {
              if (all(is.na(x))) return(NA)
              max(x, na.rm=TRUE)
            })
    censored.obs[is.na(censored.obs)] <- Tmax
    replacements <- rowSums(!is.na(history))
    generated.censored.obs <-
      sapply(1:nrow(history),
             function(i) {
               if (times[i,1] > Tmax) return(times[i,1])
               return(times[i, replacements[i] + 1])
             })
    generated.censored.obs <- generated.censored.obs[censored.obs > 0]
    censored.obs <- censored.obs[censored.obs > 0]
    tm <- c(fail.obs, censored.obs)
    tout <- c(fail.obs, generated.censored.obs)
    censored <- c(rep(FALSE, length(fail.obs)),
                  rep(TRUE, length(censored.obs)))
    rank.me <- tm * (max(tout) + 1) + tout # Split tm ties by tout
    r <- rank(rank.me, ties = "first")
#    print(cbind(times, replacements))
```

```
#    print(history)
#    print(cbind(tm, tout, censored))
#    print(Tmax)
#    browser()
  } else { # If not staggered
    ## Find the census time
    Tmax <- sort(tm)[max(2, ceiling(propFailed*numComponents))]
    ## Set censoring flag
    censored <- tm > Tmax
    r <- rank(tm, ties = "first")
    ## Set the censored values to the maximum observation
    tm[censored] <- Tmax
  }
  ## Compute ranks and adjusted ranks
  is.na(r[is.na(tm)]) <- TRUE
  ar <- rep(NA, numComponents)
  ar[!censored] <- rank(tm[!censored])
  if (pause)
    browser()
  simdata <- data.frame(
    generated = tout,
    sampleStatus = censored,
    sampleData = tm,
    overall_rank = r,
    failure_rank = ar)
  simdata = simdata[sort(simdata[,"generated"],
    index.return = TRUE)$ix, ]
  return(simdata)
  }

(dat <- weibull.staggered(
  numComponents = 10,
  propFailed = 0.8,
  shape = 3,
  scale = 50,
  stagger = 1,
  niter = 3
  ))

# EXAMPLE USE OF weibull.staggered FUNCTION

(dat <- weibull.staggered(
                          numComponents = 20,
                          propFailed = 1,
                          shape = 3,
                          scale = 50,
                          stagger = FALSE
                          ))

plot(density(dat$sampleData))
```

```
(dat <- weibull.staggered(
                          numComponents = 20,
                          propFailed = 0.2,
                          shape = 3,
                          scale = 50,
                          stagger = FALSE
                          ))

(dat <- weibull.staggered(
                          numComponents = 20,
                          propFailed = 0.2,
                          shape = 3,
                          scale = 50,
                          stagger = 2
                          ))

dat <- weibull.staggered(
  numComponents = 1000,
  propFailed = 0.2,
  shape = 3,
  scale = 50,
  stagger = FALSE
  )

plot(density(dat$sampleData))

truehist(dat$sampleData, nbins = 50, ylim = c(0, 0.06))

with(dat, mle.func(time = sampleData,
                   censored = sampleStatus,
                   check = TRUE, LL = TRUE))

###

dat <- weibull.staggered(
  numComponents = 1000,
  propFailed = 0.2,
  shape = 3,
  scale = 50,
  stagger = 2
  )

plot(density(dat$sampleData))

truehist(dat$sampleData, nbins = 50)

with(dat, mle.func(time = sampleData,
                   censored = sampleStatus,
                   check = TRUE, LL = TRUE))
```

```
simdat <- weibull.staggered


# EXAMPLE USE OF MLE.FUNC
(mle.results <-
  with(dat,
       mle.func(
                time = sampleData,
                censored = sampleStatus
  )))
# mle.results





### MRR FUNCTIONS
#function to calculate adjusted rank of jth failure
rj = function(simdat){ #function to calculate adjusted rank of jth failure
        r0 = 0
        kj = simdat[simdat$sampleStatus==0, "overall_rank"]
                                        # overall rank of the jth failure
        rj = vector(length=length(kj))
        N = dim(simdat)[1]
        rj[1] = r0 + (N+1 -r0)/(N + 1 - (kj[1] - 1))
        for(i in 2:length(kj)){
        rj[i] = rj[i-1] + (N+1 -rj[i-1])/(N + 1 - (kj[i] - 1))
        }
        return(rj)
} #close rj (adjusted rank) function


b = function(simdat, verbose = FALSE){ #function for shape parameter
        N = dim(simdat)[1]
        tj = simdat[simdat$sampleStatus==0, "sampleData"]
                                        # failure times of jth component
        k = length(tj) #number of failures
        xj = log(tj)
        rj.out <- (rj(simdat) - 0.3) / (N + 0.4)
        if (verbose) cat("Probs: ", rj.out, "\n")
        vj = log( -log(1 - rj.out))
        xv = xj * vj
        kSumxjvj = k * sum(xv)
        SumxjSumvj = sum(xj) * sum(vj)
        topline = kSumxjvj - SumxjSumvj
        kSumxjsq = k * sum(xj^2)
        Sumxjallsq = sum(xj)^2
        bottomline = kSumxjsq - Sumxjallsq
        return(topline/bottomline)
} #close b (shape) function
```

```
a = function(simdat){ #function for scale parameter
        N = dim(simdat)[1]
        tj = simdat[simdat$sampleStatus==0, "sampleData"]
                                        # failure times of jth component
        k = length(tj) #number of failures
        vj = log( log( (1) / (1 - (rj(simdat) - 0.3) / (N + 0.4))))
        xj = log(tj)
        xjsq = xj^2
        xjvj = xj * vj
        kSumxjsq = k * sum(xj^2)
        Sumxjallsq = sum(xj)^2
        topline = sum(vj) * sum(xjsq) - sum(xj) * sum(xjvj)
        b <- b(simdat)
        bottomline = -b * (kSumxjsq - Sumxjallsq)
                           # bottomline calls the shape function (b)
        return(c(b, exp(topline/bottomline)))
} #close a (scale) function



mrr.func <- function(
  simdat,
  verbose = FALSE
  ) {
  results = a(simdat)
  return(results)
}


#EXAMPLE USE OF MRR.FUNC

mrr.results = mrr.func(
    simdat = dat
    )
mrr.results




## Andrew's helper functions

make.pdf <- function(label, results_table) {
  library(ggplot2)
  suppressMessages({
  out <- with(as.data.frame(results_table),
              aggregate(x = list(
                          Shape.hat.MLE = shape.est.MLE,
                          Scale.hat.MLE = scale.est.MLE,
                          Shape.hat.MRR = shape.est.MRR,
                          Scale.hat.MRR = scale.est.MRR,
```

```
                        miae.MLE = miae.MLE,
                        miae.MRR = miae.MRR,
                        miae.C = miae.C,
                        mise.MLE = mise.MLE,
                        mise.MRR = mise.MRR,
                        mise.C = mise.C,
                        c1i.MLE = c1i.MLE,
                        c2i.MLE = c2i.MLE,
                        c1a.MLE = c1a.MLE,
                        c2a.MLE = c2a.MLE,
                        c1i.MRR = c1i.MRR,
                        c2i.MRR = c2i.MRR,
                        c1a.MRR = c1a.MRR,
                        c2a.MRR = c2a.MRR,
                        c1i.C = c1i.C,
                        c2i.C = c2i.C,
                        c1a.C = c1a.C,
                        c2a.C = c2a.C,
                        c1i = c1i,
                        c2i = c2i,
                        c1a = c1a,
                        c2a = c2a
                        ),
                      by = list(
                        numComponents = numComponents,
                        propFailed = propFailed,
                        Shape = shape.true,
                        Scale = scale.true),
                      FUN = mean, na.rm = TRUE))
n <- nrow(results_table) / nrow(out)
out.s <- with(as.data.frame(results_table),
            aggregate(x = list(
                        Shape.sd.MLE = shape.est.MLE,
                        Scale.sd.MLE = scale.est.MLE,
                        Shape.sd.MRR = shape.est.MRR,
                        Scale.sd.MRR = scale.est.MRR,
                        miae.MLE.sd = miae.MLE,
                        miae.MRR.sd = miae.MRR,
                        miae.C.sd = miae.C,
                        mise.MLE.sd = mise.MLE,
                        mise.MRR.sd = mise.MRR,
                        mise.C.sd = mise.C,
                        c1i.MLE.sd = c1i.MLE,
                        c2i.MLE.sd = c2i.MLE,
                        c1a.MLE.sd = c1a.MLE,
                        c2a.MLE.sd = c2a.MLE,
                        c1i.MRR.sd = c1i.MRR,
                        c2i.MRR.sd = c2i.MRR,
                        c1a.MRR.sd = c1a.MRR,
                        c2a.MRR.sd = c2a.MRR,
                        c1i.C.sd = c1i.C,
```

```
                          c2i.C.sd = c2i.C,
                          c1a.C.sd = c1a.C,
                          c2a.C.sd = c2a.C),
                        by = list(
                          numComponents = numComponents,
                          propFailed = propFailed,
                          Shape = shape.true,
                          Scale = scale.true),
                        FUN = sd, na.rm = TRUE))
#  browser()
  out.l <- cbind(out, out.s[,5:26])
  out.w <- reshape(out.l,
                   direction = "long",
                   varying = list(5:8, 31:34),
                   v.names = c("Estimate", "SD"),
                   timevar = "Parameter",
                   times = names(out.l)[5:8],
                   drop = c("miae.MLE","miae.MRR","miae.C",
                       "mise.MLE","mise.MRR","mise.C",
                       "miae.MLE.sd","miae.MRR.sd","miae.C.sd",
                       "mise.MLE.sd","mise.MRR.sd","mise.C.sd",
                       "c1i.MLE","c2i.MLE","c1a.MLE","c2a.MLE",
                       "c1i.MLE.sd","c2i.MLE.sd","c1a.MLE.sd","c2a.MLE.sd",
                       "c1i","c2i","c1a","c2a",
                       "c1i.MRR","c2i.MRR","c1a.MRR","c2a.MRR",
                       "c1i.MRR.sd","c2i.MRR.sd","c1a.MRR.sd","c2a.MRR.sd",
                       "c1i.C","c2i.C","c1a.C","c2a.C",
                       "c1i.C.sd","c2i.C.sd","c1a.C.sd","c2a.C.sd"))
#  browser()
  out.w$actual <- out.w$Shape
  out.w$actual[substr(out.w$Parameter, 1, 5) == "Scale"] <-
    out.w$Scale[substr(out.w$Parameter, 1, 5) == "Scale"]
  out.w$CI <- out.w$SD / sqrt(n) * 1.96
  out.check <<- out.w
  pd <- position_dodge(0.1)
  out.w$rel.bias <- with(out.w, (Estimate - actual) / actual * 100)
  out.w$rel.bias.sd <- with(out.w, SD / actual * 100)
  out.w$rel.bias.ci <- out.w$rel.bias.sd / sqrt(n) * 1.96
  pdf(paste(label, "/results_", label, ".pdf", sep = ""),
      width = 12, height = 3 * length(unique(out$Scale)))
#  browser()
  try(print(qplot(x = Shape,
                  y = rel.bias,
                  data = subset(out.w,
                    Parameter %in% c("Shape.hat.MLE", "Shape.hat.MRR")),
                  group = factor(numComponents),
                  colour = factor(numComponents),
                  geom = c("point","line"),
                  position = pd,
                  xlab = "True Shape", ylab = "Relative bias (%)")
            + facet_grid(Scale ~ Parameter, scales = "free_y")
```

```
          + labs(colour = "Units")
          + geom_errorbar(aes(ymin = rel.bias - rel.bias.ci,
                              ymax = rel.bias + rel.bias.ci), width= 0.1,
                          position = pd)
#        + coord_cartesian(ylim = c(
#                           max(c(-110,
#                                 min(subset(out.w,
#                                            out.w$Parameter %in%
#                                            c("Shape.hat.MLE",
#                                              "Shape.hat.MRR"))$rel.bias))),
#                           min(c(110,
#                                 max(subset(out.w,
#                                            out.w$Parameter %in%
#                                            c("Shape.hat.MLE",
#                                              "Shape.hat.MRR"))$rel.bias))))))
      ))
  try(print(qplot(x = Shape,
            y = rel.bias,
            data = subset(out.w,
              Parameter %in% c("Scale.hat.MLE", "Scale.hat.MRR")),
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            position = pd,
            xlab = "True Shape", ylab = "Relative bias (%)")
      + facet_grid(Scale ~ Parameter, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = rel.bias - rel.bias.ci,
                          ymax = rel.bias + rel.bias.ci), width= 0.1,
                      position = pd)
      + coord_cartesian(ylim = c(
                          max(c(-110,
                                min(subset(out.w,
                                           out.w$Parameter %in%
                                           c("Scale.hat.MLE",
                                             "Scale.hat.MRR"))$rel.bias))),
                          min(c(110,
                                max(subset(out.w,
                                           out.w$Parameter %in%
                                           c("Scale.hat.MLE",
                                             "Scale.hat.MRR"))$rel.bias)))))))
#  browser()
  out.m <- reshape(out.l,
                   direction = "long",
                   varying = list(
                     9:11, 12:14,
                     35:37, 38:40,
                     c(27,27,27), c(28,28,28), c(29,29,29), c(30,30,30),
                     c(15,19,23), c(16,20,24), c(17,21,25), c(18,22,26),
                     c(41,45,49), c(42,46,50), c(43,47,51), c(44,48,52)),
                   v.names = c(
```

```
                    "MIAE","MISE",
                    "MIAE.SD","MISE.SD",
                    "C1I","C2I","C1A","C2A",
                    "C1I.hat","C2I.hat","C1A.hat","C2A.hat",
                    "C1I.SD","C2I.SD","C1A.SD","C2A.SD"),
                  timevar = "Metric",
                  times = substr(names(out)[9:11], 6, 8),
                  drop = c("Shape.hat.MLE","Shape.hat.MRR","Scale.hat.MLE",
                      "Scale.hat.MRR","Shape.sd.MLE","Shape.sd.MRR",
                      "Scale.sd.MLE","Scale.sd.MRR"))

  out.m$miae.ci <- out.m$MIAE.SD / sqrt(n) * 1.96
  out.m$mise.ci <- out.m$MISE.SD / sqrt(n) * 1.96

  out.m$c1i.ci <- out.m$C1I.SD / sqrt(n) * 1.96
  out.m$c1a.ci <- out.m$C1A.SD / sqrt(n) * 1.96
  out.m$c2i.ci <- out.m$C2I.SD / sqrt(n) * 1.96
  out.m$c2a.ci <- out.m$C2A.SD / sqrt(n) * 1.96

  out.m$c1i.bias <- out.m$C1I.hat - out.m$C1I
  out.m$c1a.bias <- out.m$C1A.hat - out.m$C1A
  out.m$c2i.bias <- out.m$C2I.hat - out.m$C2I
  out.m$c2a.bias <- out.m$C2A.hat - out.m$C2A

  out.m$c1i.rmse <- sqrt(out.m$c1i.bias^2 + out.m$C1I.SD^2)
  out.m$c1a.rmse <- sqrt(out.m$c1a.bias^2 + out.m$C1A.SD^2)
  out.m$c2i.rmse <- sqrt(out.m$c2i.bias^2 + out.m$C2I.SD^2)
  out.m$c2a.rmse <- sqrt(out.m$c2a.bias^2 + out.m$C2A.SD^2)

#  browser()

  try(print(qplot(x = Shape,
            y = MIAE,
            data = out.m,
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            position = pd,
            xlab = "True Shape", ylab = "MIAE")
       + facet_grid(Scale ~ Metric, scales = "free_y")
       + labs(colour = "Units")
       + geom_errorbar(aes(ymin = MIAE - miae.ci,
                           ymax = MIAE + miae.ci), width= 0.1, position = pd)))
  try(print(qplot(x = Shape,
            y = MISE,
            data = out.m,
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            position = pd,
            xlab = "True Shape", ylab = "MISE")
```

```
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = MISE - mise.ci,
                          ymax = MISE + mise.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c1i.bias,
            data = out.m,
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            main = "1e-6 Hazard (Hours)",
            position = pd,
            xlab = "True Shape", ylab = "Bias")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c1i.bias - c1i.ci,
                          ymax = c1i.bias + c1i.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c2i.bias,
            data = out.m,
            group = factor(numComponents),
            main = "1e-5 Hazard (Hours)",
            colour = factor(numComponents),
            geom = c("point","line"),
            position = pd,
            xlab = "True Shape", ylab = "Bias")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c2i.bias - c2i.ci,
                          ymax = c2i.bias + c2i.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c1a.bias,
            data = out.m,
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            main = "1e-6 Average Hazard (Hours)",
            position = pd,
            xlab = "True Shape", ylab = "Bias")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c1a.bias - c1a.ci,
                          ymax = c1a.bias + c1a.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c2a.bias,
            data = out.m,
            group = factor(numComponents),
```

```
            colour = factor(numComponents),
            geom = c("point","line"),
            position = pd,
            main = "1e-5 Average Hazard (Hours)",
            xlab = "True Shape", ylab = "Bias")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c2a.bias - c2a.ci,
                          ymax = c2a.bias + c2a.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c1i.rmse,
            data = out.m,
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            main = "1e-6 Hazard (Hours)",
            position = pd,
            xlab = "True Shape", ylab = "RMSE")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c1i.rmse - c1i.ci,
                          ymax = c1i.rmse + c1i.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c2i.rmse,
            data = out.m,
            group = factor(numComponents),
            main = "1e-5 Hazard (Hours)",
            colour = factor(numComponents),
            geom = c("point","line"),
            position = pd,
            xlab = "True Shape", ylab = "RMSE")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c2i.rmse - c2i.ci,
                          ymax = c2i.rmse + c2i.ci), width= 0.1, position = pd)))

try(print(qplot(x = Shape,
            y = c1a.rmse,
            data = out.m,
            group = factor(numComponents),
            colour = factor(numComponents),
            geom = c("point","line"),
            main = "1e-6 Average Hazard (Hours)",
            position = pd,
            xlab = "True Shape", ylab = "RMSE")
      + facet_grid(Scale ~ Metric, scales = "free_y")
      + labs(colour = "Units")
      + geom_errorbar(aes(ymin = c1a.rmse - c1a.ci,
                          ymax = c1a.rmse + c1a.ci), width= 0.1, position = pd)))
```

```
  try(print(qplot(x = Shape,
              y = c2a.rmse,
              data = out.m,
              group = factor(numComponents),
              colour = factor(numComponents),
              geom = c("point","line"),
              position = pd,
              main = "1e-5 Average Hazard (Hours)",
              xlab = "True Shape", ylab = "RMSE")
        + facet_grid(Scale ~ Metric, scales = "free_y")
        + labs(colour = "Units")
        + geom_errorbar(aes(ymin = c2a.rmse - c2a.ci,
                              ymax = c2a.rmse + c2a.ci), width= 0.1, position = pd)))

  dev.off()
})
}



simulate <- function(label,
                     numSim = 1000,
                     numComponents = c(1000, 500, 100, 50, 20),
                     propFailed = c(0.2),
                     shape = c(0.5, 1.5, 3.0, 4.5, 6.0),
                     scale = c(50),
                     stagger = FALSE,
                     results = FALSE,
                     verbose = FALSE,
                     pdf = TRUE,
                     niter = 5,
                     mc = 4,
                     seed = 2115153) {
    dir.create(label)
    results_table <- expand.grid(
        numSim = 1:numSim,
        numComponents = numComponents,
        propFailed = propFailed,
        shape.true = shape,
        scale.true = scale,
        stagger = stagger)
    results_table$index <- 1:nrow(results_table)
#  cat(dim(results_table), "\n")
#    res.out <- foreach(i = 1:nrow(results_table), .combine = 'rbind') %dopar% {
#    res.out <- foreach(i = 1:nrow(results_table), .combine = 'rbind') %do% {
    sim.fun <-
        function(i) {
            set.seed(i * 1024 + seed) # Work-around for parallel
            dat <- with(results_table[i,],
                        weibull.staggered(
```

```
                         numComponents = numComponents,
                         propFailed = propFailed,
                         shape = shape.true,
                         scale = scale.true,
                         stagger = stagger,
                         niter = niter))
         mle <- with(dat, mle.func(time = sampleData,
                                   censored = sampleStatus,
                                   p.start = c(
                                       shape = results_table$shape.true[i],
                                       scale = results_table$scale.true[i]),
                                   LL = TRUE))
         mle.ie <-
             ie(mle[2:1],
                  c(results_table$scale.true[i],
                    results_table$shape.true[i]))
         mle.cutoffs <- hazard.cutoffs(mle[2:1])
         mrr <- mrr.func(dat)
         mrr.ie <-
             ie(mrr[2:1],
                  c(results_table$scale.true[i],
                    results_table$shape.true[i]))
         comb.ie <-
             ie(c(mle[2], mrr[1]),
                  c(results_table$scale.true[i],
                    results_table$shape.true[i]))
         mrr.cutoffs <- hazard.cutoffs(mrr[2:1])
         comb.cutoffs <- hazard.cutoffs(c(mle[2], mrr[1]))
         out <- c(i, mle, mle.ie, mle.cutoffs,
                  mrr, mrr.ie, mrr.cutoffs,
                  comb.ie, comb.cutoffs)
#         if (length(out) > 20) browser()
         if (verbose & (ceiling(i/10) == i/10)) cat(i, " ")
#         browser()
         out
     }
 if (mc > 1) {
     library(parallel)
     cat(paste("Going parallel: ", detectCores(),
               " cores detected; ", mc, "used.\n"))
     cat(paste("Sims: ", nrow(results_table), "\n"))
     my_cluster <- makeForkCluster(mc)
     sim.out <- parLapply(my_cluster, 1:nrow(results_table), sim.fun)
     stopCluster(my_cluster)
 } else {
     cat("Serial execution.\n")
     cat(paste("Sims: ", nrow(results_table), "\n"))
     sim.out <- lapply(1:nrow(results_table), sim.fun)
 }
 if (verbose) cat("Finished simulations\n")
 if (verbose) cat("Length: ", length(sim.out), "\n")
```

```
    if (verbose) cat("Mean lengths: ", mean(sapply(sim.out, length)), "\n")
#    browser()
    res.out <- as.data.frame(do.call(rbind, sim.out))
    if (ncol(res.out) > 50) browser()
    if (verbose) cat("Res.out\n", dim(res.out), "\n")
    colnames(res.out) <-
        c("index",
          "shape.est.MLE", "scale.est.MLE", "ML", "miae.MLE", "mise.MLE",
          "c1i.MLE", "c2i.MLE", "c1a.MLE", "c2a.MLE",
          "shape.est.MRR", "scale.est.MRR", "miae.MRR", "mise.MRR",
          "c1i.MRR", "c2i.MRR", "c1a.MRR", "c2a.MRR",
          "miae.C", "mise.C","c1i.C", "c2i.C", "c1a.C", "c2a.C")
#    browser()
    results_table <- merge(results_table, res.out)
    if (verbose) cat("Results\n", dim(results_table), "\n")
    cuts <- unique(results_table[, c("scale.true","shape.true")])
    true.cutoffs <-
        lapply(1:nrow(cuts), function(x) hazard.cutoffs(c(cuts$scale.true[x],
                                                          cuts$shape.true[x])))
    true.cutoffs <- cbind(cuts, as.data.frame(do.call(rbind, true.cutoffs)))
    names(true.cutoffs)[3:6] <- c("c1i", "c2i", "c1a", "c2a")
#    browser()
    if (verbose) cat("Cuts\n", dim(true.cutoffs), "\n")
    results_table <- merge(results_table, true.cutoffs, all.x = TRUE)
    if (pdf) {
        make.pdf(label, results_table)
        mail("andrew_2f66@sendtodropbox.com",
           paste("DSTO_", label, sep = ""),
             "All done.",
             paste(label, "/results_", label, ".pdf", sep = ""))
    }
    if (results) {
      save(results_table,
           file = paste(label, "/results_", label, ".RData", sep = ""))
        mail("andrew_2f66@sendtodropbox.com",
           paste("DSTO_", label, sep = ""),
             "All done.",
             paste(label, "/results_", label, ".RData", sep = ""))
    }
    mail("mensurationist@gmail.com",
         paste(label, "completed."),
         "All done.")
#    browser()
    return(invisible(results_table))
}


nloglik <- function(p, time, censored) {
  nll <- -sum(ifelse(censored,
                     suppressWarnings(pweibull(time, p[1], p[2],
                                               lower.tail = FALSE,
```

```
                                              log.p = TRUE)),
                    suppressWarnings(dweibull(time, p[1], p[2],
                                              log = TRUE))))
  if(is.finite(nll)) nll else .Machine$double.xmax
}


mise <- function(estimate, process, upp = Inf, grain = 1000) {
#    answer <- try(integrate(function(x) {
#        (dweibull(x, scale = estimate[1], shape = estimate[2]) -
#         dweibull(x, scale = process[1], shape = process[2]))^2
#    },
#                     subdivisions = 1000L,
#                     lower = 0.5, upper = upp)$value,
#               silent = TRUE)
#    if (class(answer) == "try-error") {
  cutoffs <- qweibull((1:grain)/(1+grain),
                     scale = process[1],
                     shape = process[2])
  quant.est <- pweibull(cutoffs, scale = estimate[1], shape = estimate[2])
  diff.est <- quant.est[2:(1+grain)] - quant.est[1:grain]
  quant.pro <- pweibull(cutoffs, scale = process[1], shape = process[2])
  diff.pro <- quant.pro[2:(1+grain)] - quant.pro[1:grain]
  answer <- sum((diff.est - diff.pro)^2)
#    } else return(answer)
  return(answer)
}

# mise <- function(estimate, process, upp = Inf) 0

miae <- function(estimate, process, grain = 1000000) {
#    answer <- try(integrate(function(x) {
#        abs(dweibull(x, scale = estimate[1], shape = estimate[2]) -
#            dweibull(x, scale = process[1], shape = process[2]))
#      },
#                            subdivisions = 1000L,
#                            lower = 0.5, upper = upp)$value,
#                  silent = TRUE)
#    if (class(answer) == "try-error") {
  cutoffs <- qweibull((1:grain)/(1+grain),
                     scale = process[1], shape = process[2])
  quant.est <- pweibull(cutoffs, scale = estimate[1], shape = estimate[2])
  diff.est <- quant.est[2:grain] - quant.est[1:(grain-1)]
  quant.pro <- pweibull(cutoffs, scale = process[1], shape = process[2])
  diff.pro <- quant.pro[2:grain] - quant.pro[1:(grain-1)]
  answer <- sum(abs(diff.est - diff.pro))
#      answer <- sum(abs(diff.est - diff.pro))
#    } else
  return(answer)
#    return (c(answer, answer2))
  }
```

```
ie <- function(estimate, process, grain = 1e6) {
  cutoffs <- qweibull((1:grain)/(1+grain),
                      scale = process[1], shape = process[2])
#  diffs <- cutoffs[2:grain] - cutoffs[1:(grain-1)]
  quant.est <- pweibull(cutoffs, scale = estimate[1], shape = estimate[2])
  diff.est <- quant.est[2:grain] - quant.est[1:(grain-1)]
  quant.pro <- pweibull(cutoffs, scale = process[1], shape = process[2])
  diff.pro <- quant.pro[2:grain] - quant.pro[1:(grain-1)]
  miae <- sum(abs(diff.est - diff.pro))
  mise <- sum((diff.est - diff.pro)^2)
  return(c(miae, mise))
  }

# miae <- function(estimate, process, upp = Inf) 0

ie(estimate = c(96810.7, 4.74875), process = c(100000, 6))

ie(estimate = c(96810.7, 4.74875), process = c(100000, 6), 1e5)



miae(estimate = c(96810.7, 4.74875), process = c(100000, 6))

mise(c(55, 2.5), c(50, 3))

mise(c(14.766667, 0.4982703), c(50, 0.5), 1000)

miae(c(14.766667, 0.4982703), c(50, 0.5), 1000)

mise(c(14.766666, 0.4982704), c(50, 0.5))

mise(c(14.766666, 0.5), c(50, 0.5))

mise(c(14.766667, 0.5), c(50, 0.5))

mise(c(14.766668, 0.5), c(50, 0.5))

mise(c(15, 0.5), c(50, 0.5))

#curve((dweibull(x, scale = 14.7666674, shape = 0.4982703) -
#       dweibull(x, scale = 50, shape = 0.5))^2,
#      from = 0, to = 10, n = 101)

#curve(dweibull(x, scale = 14.7666674, shape = 0.4982703),
#      from = 0, to = 1000, n = 1001)
#curve(dweibull(x, scale = 50, shape = 0.5),
#      from = 0, to = 1000, n = 1001, add=TRUE)

instant.hazard.cutoffs <- function(params, upper = 10000) {
```

```
  out <- try({
      quants <- pweibull(1:upper, scale = params[1], shape = params[2])
      diffs <- quants[2:upper] - quants[1:(upper-1)]
      first <- min(which(diffs > 1e-6))
      second <- min(which(diffs > 1e-5))
      c(first, second)},
             silent = TRUE)
  if (class(out) == "try-error") return(c(NA,NA)) else return(out)
}

instant.hazard.cutoffs(c(3000, 3))


average.hazard.cutoffs <- function(params, upper = 10000) {
    out <- try({
        ahc <- function(x, cutoff) {
            -log(1 - pweibull(x, scale = params[1], shape = params[2])) / x - cutoff
        }
        first <- uniroot(ahc, interval = c(1, upper), cutoff = 1e-6)$root
        second <- uniroot(ahc, interval = c(1, upper), cutoff = 1e-5)$root
        c(first, second)},
               silent = TRUE)
    if (class(out) == "try-error") return(c(NA,NA)) else return(out)
}

average.hazard.cutoffs(c(3000, 3))


hazard.cutoffs <- function(params, cutoff = NULL) {
  my.scale <- params[1]; my.shape <- params[2]
  if (is.null(cutoff))
    if (my.shape > 1) {
      max.cutoff <-
        ceiling(my.scale * ((my.shape - 1) / my.shape) ^ (1 / my.shape))
    } else {
      max.cutoff <- my.scale
    }
  grain <- 1:150000
  quants <- pweibull(grain, scale = my.scale, shape = my.shape)
  diffs <- quants[grain[-1]] - quants[grain[-length(grain)]]
  out.1 <- try(min(which(diffs > 1e-6)))
  out.2 <- try(min(which(diffs > 1e-5)))
  ahc <- cumsum(diffs) / 1:length(diffs)
  out.3 <- try(min(which(ahc > 1e-6)))
  out.4 <- try(min(which(ahc > 1e-5)))
  if (class(out.1) == "try-error") {
      out.1 <- 1; is.na(out.1) <- TRUE
  }
  if (class(out.2) == "try-error") {
      out.2 <- 1; is.na(out.2) <- TRUE
  }
```

```
  if (class(out.3) == "try-error") {
      out.3 <- 1; is.na(out.3) <- TRUE
  }
  if (class(out.4) == "try-error") {
      out.4 <- 1; is.na(out.4) <- TRUE
  }
  out <- c(out.1, out.2, out.3, out.4)
  is.na(out) <- is.infinite(out)
  return(out)
}

hazard.cutoffs(c(3000, 3))

hazard.cutoffs(c(100000, 3))

# hazard.cutoffs <- function(params, cutoff = NULL) rep(0, 4)

mail <- function(address, subject, message, attach = NULL) {
  if (is.null(attach)) {
    system(paste("echo '", message,
                 "' | mutt -s '", subject,
                 "' ", address, sep=""))
  } else {
    system(paste("echo '", message,
                 "' | mutt -s '", subject, "'",
                 paste(" -a ", attach, collapse=""),
                 " ", address, sep=""))
  }
}

weibull.mode <- function(params) {
  my.scale <- params[1]; my.shape <- params[2]
  dweibull(my.scale * ((my.shape - 1) / my.shape) ^ (1 / my.shape),
           scale = my.scale, shape = my.shape)
}


.ls.objects <- function (pos = 1, pattern, order.by,
                         decreasing=FALSE, head=FALSE, n=5) {
    napply <- function(names, fn) sapply(names, function(x)
                                         fn(get(x, pos = pos)))
    names <- ls(pos = pos, pattern = pattern)
    obj.class <- napply(names, function(x) as.character(class(x))[1])
    obj.mode <- napply(names, mode)
    obj.type <- ifelse(is.na(obj.class), obj.mode, obj.class)
    obj.size <- napply(names, object.size)
    obj.dim <- t(napply(names, function(x)
                        as.numeric(dim(x))[1:2]))
    vec <- is.na(obj.dim)[, 1] & (obj.type != "function")
    obj.dim[vec, 1] <- napply(names, length)[vec]
    out <- data.frame(obj.type, obj.size, obj.dim)
```

```
    names(out) <- c("Type", "Size", "Rows", "Columns")
    if (!missing(order.by))
        out <- out[order(out[[order.by]], decreasing=decreasing), ]
    if (head)
        out <- head(out, n)
    out
}
                                        # shorthand
lsos <- function(..., n=10) {
    .ls.objects(..., order.by="Size", decreasing=TRUE, head=TRUE, n=n)
}
```

A.2 2_tests.R

```
# source functions

source('1_Functions.R')

test <- simulate(label = "Compare_S50_F02_StF_stupid",
                 numSim = 2,
                 numComponents = c(20, 1000),
                 propFailed = 1,
                 shape = c(0.5, 6),
                 scale = c(50, 100000),
                 stagger = FALSE,
                 results = FALSE,
                 verbose = TRUE,
                 pdf = TRUE,
                 mc = 1)

test <- simulate(label = "Compare_S50_F02_StF_test",
                 numSim = 2,
                 numComponents = c(100, 1000),
                 propFailed = 1,
                 shape = c(0.5, 6),
                 scale = c(50, 100000),
                 stagger = FALSE,
                 results = FALSE,
                 verbose = TRUE,
                 pdf = TRUE,
                 mc = 4)


test <- simulate(label = "Compare_S50_F02_StF_test",
                 numSim = 5,
                 numComponents = c(20, 1000),
                 propFailed = 0.2,
                 shape = c(0.5, 6),
                 scale = c(50, 100000),
                 stagger = FALSE,
```

```
                  results = TRUE,
                  verbose = TRUE,
                  mc = 1)
```

A.3 3_simulate.R

```
# source functions

source('1_Functions.R')

numComponents <- c(20, 50, 100, 500, 1000)
shapes <- c(0.5, 0.75, 1, 1.25, 1.5, 3, 4.5, 6)
scales <- c(500, 2500, 5000, 50000)

(warm_up <-
  system.time(simulate(label = "Warm_Up",
                      numSim = 4,
                      propFailed = 0.02,
                      numComponents = c(20, 1000),
                      shape = c(0.5, 6),
                      scale = c(500, 50000),
                      stagger = 0,
                      niter = 5,
                      results = TRUE,
                      pdf = TRUE,
                      mc = 4)))

## 100%, 90%, 70%, 50%, 20%, 5% and 2%.  Stagger = 0, 1, 2, 3

times <- vector(length = 28, mode = "list")
i <- 1

for (stag in 0) {
  for (fail.prop in c(0.02, 0.05, 0.2, 0.5, 0.7, 0.9, 1)) {
    (times[[i]] <-
     system.time(simulate(label = paste(
                          "Stagger-", stag, "_",
                          "Fail-", ceiling(fail.prop * 100), "_",
                          "Sim-1000", sep = ""),
                        numSim = 1000,
                        propFailed = fail.prop,
                        numComponents = numComponents,
                        shape = shapes,
                        scale = scales,
                        stagger = stag,
                        results = TRUE,
                        pdf = TRUE,
                        niter = 5,
                        mc = 30)))
    i <- i + 1
```

```
  }
}

for (stag in 3) {
  for (fail.prop in c(0.7, 0.9, 1)) {
    (times[[i]] <-
     system.time(simulate(label = paste(
                           "Stagger-", stag, "_",
                           "Fail-", ceiling(fail.prop * 100), "_",
                           "Sim-1000", sep = ""),
                         numSim = 1000,
                         propFailed = fail.prop,
                         numComponents = numComponents,
                         shape = shapes,
                         scale = scales,
                         stagger = stag,
                         results = TRUE,
                         pdf = TRUE,
                         niter = 5,
                         mc = 30)))
    i <- i + 1
  }
}

for (stag in 2) {
  for (fail.prop in c(0.02, 0.05, 0.2, 0.5, 0.7, 0.9, 1)) {
    (times[[i]] <-
     system.time(simulate(label = paste(
                           "Stagger-", stag, "_",
                           "Fail-", ceiling(fail.prop * 100), "_",
                           "Sim-1000", sep = ""),
                         numSim = 1000,
                         propFailed = fail.prop,
                         numComponents = numComponents,
                         shape = shapes,
                         scale = scales,
                         stagger = stag,
                         results = TRUE,
                         pdf = TRUE,
                         niter = 5,
                         mc = 30)))
    i <- i + 1
  }
}

## Special case

fail.prop <- 0.01
numComponents <- 500

for (stag in 0:3) {
```

```
    special <- system.time(simulate(label = paste(
                                      "Stagger-", stag, "_",
                                      "Fail-", ceiling(fail.prop * 100), "_",
                                      "Sim-1000", sep = ""),
                                    numSim = 1000,
                                    propFailed = fail.prop,
                                    numComponents = numComponents,
                                    shape = shapes,
                                    scale = scales,
                                    stagger = stag,
                                    results = TRUE,
                                    pdf = TRUE,
                                    niter = 5,
                                    mc = 20))
}
```

B Appendix: Annotated Bibliography

B.1 Examples of failure analysis

This section provides a brief overview of papers that address analysis of filaures in engines, turbines, and fleets of vehicles, whether using the Weibull distribution or not.

B.2 Example applications of the 2-parameter Weibull distribution

This section provides an overview of the application of the 2-parameter Weibull distribution in science and management.

Adaramola, M. S., O. M. Oyewola, O. S. Ohunakin, and O. O. Akinnawonu (2014). Performance evaluation of wind turbines for energy generation in Niger Delta, Nigeria. *Sustainable Energy Technologies and Assessments 6*, 75–85.

> Evaluates wind energy potentials of seven selected locations spreading across Niger-Delta region of Nigeria using wind speed data, and subjected to 2-parameter Weibull distribution functions.

Allsopp, P. (1981). Development, longevity and fecundity of the false wireworms *Pterohelaeus darlingensis* and *P. alternatus* (Coleoptera: Tenebrionidae). I. effect of constant temperature. *Australian Journal of Zoology 29*, 605–619.

> False Wireworms were reared, and weekly data on fecundity and survival of adults was collected. Adult survivorship data is summarized by means of the Weibull frequency distribution (Pinder et al. (1978) *q.v.*).

Amarasekare, P. and R. Sifuentes (2012). Elucidating the temperature response of survivorship in insects. *Functional Ecology 26*, 959–968.

> We use a life-history-based approach to elucidate the temperature response of cumulative (egg-to-adult) survivorship. We show that the temperature response of cumulative survivorship depends on whether or not different life-history stages/age classes respond differentially to temperature. We can describe the relationship between the instantaneous risk of death and age using the hazard function of the Weibull distribution. The Weibull distribution provides a useful tool for analysing survivorship data because the shape and scale parameters summarize all the survivorship information in a life table (Pinder et al. (1978), *q.v.*).

Araki, J., H. Matsubara, J. Shimizu, T. Mikane, S. Mohri, J. Mizuno, M. Takaki, T. Ohe, M. Hirakawa, and H. Suga (1999). Weibull distribution function for cardiac contraction: integrative analysis. *American Journal of Physiology—Heart and Circulatory Physiology 277*, H1940–H1945.

> The Weibull distribution is widely used to analyze the cumulative loss of performance, i.e., breakdown, of a complex system in systems engineering. We found for the first time that the difference curve of two Weibull distribution functions almost identically fitted the isovolumically contracting left ventricular (LV) pressure-time curve. This Weibull-function model of the heart seems to give a new systems engineering or integrative physiological view of the logic underlying LV isovolumic pressure generation.

Arfeen, M. A., K. Pawlikowski, D. McNickle, and A. Willig (2012). Towards a combined traffic modeling framework for access and core networks. In *ATNAC 2012: 9th International Australasian Telecommunication Networks and Applications Conference.*

Both Long Range Dependence (LRD) and Short Range Dependence (SRD) co-exist in Internet traffic. We review the transformation between LRD and SRD at different levels of traffic superposition (ISP tiers) and propose a simple combined framework for traffic modeling for source, access and core (backbone) networks. This flow level framework emphasizes the role of Pareto distribution in source modeling; and, Weibull distribution in access and core network traffic modeling.

Arfeen, M. A., K. Pawlikowski, D. McNickle, and A. Willig (2013). The role of the Weibull distribution in Internet traffic modeling. In *Proceedings of the 2013 25th International Teletraffic Congress.*

This paper highlights the important role played by the two parameter Weibull distribution in Internet traffic modeling.

Bahi, J., M. Haddad, M. Hakem, and H. Kheddouci (2014). Efficient distributed lifetime optimization algorithm for sensor networks. *Ad Hoc Networks 16*, 1–12.

One of the main design challenges in Wireless Sensor Networks (WSN) is to prolong the system lifetime, while achieving acceptable quality of service for applications. In this paper, we introduce an efficient lifetime optimization and self-stabilizing algorithm to enhance the lifetime of wireless sensor networks especially when the reliabilities of sensor nodes are expected to decrease due to use and wear-out effects.

Bailey, R. L. and T. Dell (1973). Quantifying diameter distributions with the Weibull function. *Forest Science 19*, 97–104.

The Weibull probability density function is proposed as a diameter distribution model for stands of trees. Its advantages include flexibility in shape and simplicity of mathematical derivations. Estimation and interpretation of parameters are discussed and illustrated with published data. apabilities. Simplicity of algebraic manipulations and ability to assume a variety of curve shapes should make the Weibull useful for other biological models.

Bernard, A. and E. Bosi-Levenbach (1953). The plotting of observations on probability paper. *Stat. Neederlandica 7*, 163–173.

Celik, A. N. (2003). Energy output estimation for small-scale wind power generators using Weibull-representative wind data. *Journal of Wind Engineering and Industrial Aerodynamics 91*, 693–707.

Estimation of energy output for small-scale wind power generators is the subject of this article. Monthly wind energy production is estimated using the Weibull-representative wind data from 5 different locations. The Weibull parameters are determined based on the wind distribution statistics calculated from the measured data, using the gamma function. It is shown that the Weibull-representative data estimate the wind energy output very accurately.

Celik, A. N. (2006). A simplified model for estimating yearly wind fraction in hybrid-wind energy systems. *Renewable Energy 31*, 105–118.

This article presents a simplified algorithm to estimate the yearly wind fraction, the fraction of energy demand provided by wind generator, in a hybrid-wind system (typically a PV-wind) with battery storage. The novel method correlates the yearly wind fraction with the parameters of the Weibull distribution function, thus, offering a general methodology. The yearly wind fraction curves are mathematically represented using a 2-parameter model.

Cockfield, S., S. Butkewich, K. Samoil, and D. Mahr. (1994). Forecasting flight activity of *Sparganothis sulfureana* (Lepidoptera: Tortricidae) in cranberries. *Journal of Economic Entomology 87*, 193–196.

Connolly, S. R. and A. H. Baird (2010). Estimating dispersal potential for marine larvae: dynamic models applied to scleractinian corals. *Ecology 91*, 3572–3583.

Dispersal influences ecological dynamics, evolution, biogeography, and biodiversity conservation, but models of larval dispersal in marine organisms make simplifying assumptions that are likely to approximate poorly the temporal dynamics of larval survival and capacity for settlement. To improve upon these assumptions, we here develop simple models of dispersal potential, and fit these models to empirical competence and survival data for five scleractinian coral species. The models fit the data well, incorporating qualitative features of competence and survival that traditional approaches to modeling dispersal do not.

Cooley, P. C., L. E. Myers, and D. N. Hamill (1996). A meta-analysis of estimates of the AIDS incubation distribution. *European Journal of Epidemiology 12*, 229–235.

The paper investigates several statistical distributions that might describe the distribution of HIV/AIDS incubation, namely the Weibull, gamma, log-logistic and lognormal.

Ewbank, D. C. (2002). A multistate model of the genetic risk of Alzheimer's disease. *Experimental Aging Research 28*, 477–499.

This study models the effect of heterogeneity of risk on the age pattern of incidence of Alzheimer's disease.

FAA (2003). Advisory circular: Continued airworthiness assessments of powerplant and auxiliary power unit installations of transport category airplanes. Technical report, Federal Aviation Administration, U.S. Department of Transportation. 39-8.

The document describes acceptable means, but not the only means, for demonstrating compliance with the applicable regulations.

Gorter, W., J. H. van Angelen, J. T. M. Lenaerts, and M. R. van den Broeke (2014). Present and future near-surface wind climate of Greenland from high resolution regional climate modelling. *Climate Dynamics 42*, 1595–1611.

The present and twenty-first century near-surface wind climate of Greenland is presented using output from the regional atmospheric climate model RACMO2. The Weibull shape parameter is used to classify the wind climate.

Gotoh, T., M. Fukuhara, and K.-I. Kikuchi (2008). Mathematical model for change in diameter distribution of baculovirus-infected Sf-9 insect cells. *Biochemical Engineering Journal 40*, 379–386.

Sf-9 insect cells were examined for size before and after baculovirus infection. The distribution of the diameter of uninfected Sf-9 insect cells was described by the normal distribution function. A mathematical model was developed to simulate the size distribution of virus-infected Sf-9 insect cells at any given time after viral infection. (A cumulative Weibull distribution function was used to describe virus amplification by the infected cells.)

Grissino-Mayer, H. D. (1999). Modeling fire interval data from the American Southwest with the Weibull distribution. *International Journal of Wildland Fire 9*, 37–50.

In this study, the Weibull distribution is tested as a possible model for fire interval data derived from dendrochronologically-dated fire scars from four sites in the American Southwest. Two- and three-parameter Weibull distributions were fit to fire interval data sets.

Higgins, S. I. and D. M. Richardson (1999). Predicting plant migration rates in a changing world: The role of long-distance dispersal. *American Naturalist 153*, 464–475.

Mixtures of Weibull distributions to dispersal data—highlighting that models of plant migration severely underestimate rates of spread. Models of plant migration based on estimates of biological parameters severely underestimate the rate of spread when compared to empirical estimates of plant migration rates. We fit mixtures of Weibull distributions to several dispersal data sets and show that statistical and biological criteria for selecting the most appropriate statistical model conflict.

Jin, J., W. Wang, G. Wets, X. Wang, Y. Mao, and X. Jiang (2014). Effect of restricted sight on right-turn driver behavior with pedestrians at signalized intersection. *Advances in Mechanical Engineering 2014*, 1–6. Article ID 565394.

When right-turning vehicles on divided lanes at an intersection share the same phase with pedestrians, medium size vehicles beside the turning vehicles block the view of drivers and pedestrians. This paper analyzes lag/gap acceptance behavior with/without sight obstruction and the headway of following right-turn vehicles. The cumulative Weibull distribution function was used to estimate parameters of lag/gap acceptance probabilities.

Jácome, A. A. A., D. R. Wohnrath, C. S. Neto, E. C. Carneseca, S. V. Serrano, L. S. Viana, J. S. Nunes, E. Z. Martinez, and J. S. Santos (2014). Prognostic value of epidermal growth factor receptors in gastric cancer: a survival analysis by Weibull model incorporating long-term survivors. *Gastric Cancer 17*, 76–86.

Tissues from gastric tumors were analyzed. Correlations between receptor expression and clinicopathological characteristics were performed according to the chi-square test. Survival analysis was calculated according to the Weibull model with a mixture model incorporating long-term survivors. Multivariate analysis of prognostic factors was performed by a regression model incorporating long-term survivors with the Weibull distribution.

Kim, D.-S., J.-H. Lee, and M.-S. Yiem (2001). Temperature-dependent development of *Carposina sasakii* (Lepidoptera: Carposinidae) and its stage emergence models. *Environmental Entomology 30*, 298–305.

Carposina sasakii is the most destructive insect pest of fruit trees such as apple, peach, and pear in Korea. Temperature-dependent development of *C. sasakii* was studied, and models developed to simulate the proportion of individuals shifted from one stage to the next. The models, constructed using the modified Sharpe and DeMichele model and the two-parameter Weibull function, predicted the patterns of stage emergences in the field.

Kose, F., M. H. Aksoy, and M. Ozgoren (2014). An assessment of wind energy potential to meet electricity demand and economic feasibility in Konya, Turkey. *International Journal of Green Energy 11*, 559–576.

In this study, wind energy potential of Selcuk University campus region was investigated by means of wind data, which were obtained locally from a special observation station. For consideration of the calculation, wind speed frequency histogram, Rayleigh and Weibull distributions, wind direction and temperature data were also used.

Lun, I. Y. and J. C. Lam (2000). A study of Weibull parameters using long-term wind observations. *Renewable Energy 20*, 145–153.

The two parameters of a Weibull density distribution function were calculated for three different locations in Hong Kong. Based on these data, it was found that the numerical values of the shape and scale parameters for these weather stations varied over a wide range.

Lázaro, E., C. Escarmís, J. Pérez-Mercader, S. C. Manrubia, and E. Domingo (2003). Resistance of virus to extinction on bottleneck passages: Study of a decaying and fluctuating pattern of fitness loss. *Proceedings of the National Academy of Sciences 100*, 10830–10835.

Here we report an analysis of fitness evolution of several low fitness foot-and-mouth disease virus clones subjected to 50 plaque-to-plaque transfers. Unexpectedly, fitness decrease (measured here as diminished capacity to produce infectious progeny), rather than being continuous and monotonic, displayed a fluctuating pattern, which was influenced by both the virus and the state of the host cell as shown by effects of recent cell passage history. The probability of fitness values in the evolving bottlenecked populations fit a Weibull distribution.

Maynar, P. and E. Trizac (2011). Entropy of continuous mixtures and the measure problem. *Physical Review Letters 106*, 160603.

In its continuous version, the entropy functional measuring the information content of a given probability density may be plagued by a "measure" problem that results from improper weighting of phase space. This issue is addressed considering a generic collision process whereby a large number of particles/agents randomly and repeatedly interact in pairs, with prescribed conservation law(s). We find a sufficient condition under which the stationary single particle distribution function maximizes an entropy-like functional, that is free of the measure problem. This condition amounts to a factorization property of the Jacobian associated to the binary collision law, from which the proper weighting of phase space directly follows.

Moritz, M. A. (2003). Spatiotemporal analysis of controls on shrubland fire regimes: age dependency and fire hazard. *Ecology 84*, 351–361.

Large fires in shrublands of California are widely attributed to decades of fire suppression. Using mapped fire history data, I analyzed burning patterns using a geographic information system. To estimate the degree of age dependency exhibited by the fire regime at different spatial scales, I employed methods of fire frequency analysis (i.e., fitting a generalized Weibull function to fire interval distributions). Results indicated that shrubland fires have frequently burned through young age classes of vegetation, exhibiting a minimal degree of age dependency.

Nedaei, M. (2014). Wind resource assessment in Hormozgan province in Iran. *International Journal of Sustainable Energy 33*, 650–694.

Statistical analysis of wind data is performed using the Weibull distribution function to determine the potential of wind power in two locations in Hormozgan province.

Papadopoulou, V., K. Kosmidis, M. Vlachou, and P. Macheras (2006). On the use of the Weibull function for the discernment of drug release mechanisms. *International Journal of Pharmaceutics 309*, 44–50.

In this study, drug release kinetics of various published data and experimental data from commercial or prepared controlled release formulations of diltiazem and diclofenac are analyzed using the Weibull function. It provides experimental evidence for the successful use of the Weibull function in drug release studies.

Pinder, III, J. E., J. G. Wiener, and M. H. Smith (1978). The Weibull distribution: a new method of summarizing survivorship data. *Ecology 59*, 175–179.

Survivorship data can be effectively summarized using the shape and scale parameters of the Weibull frequency distribution. The shape parameter controls the rate of change of the age specific mortality rate and, therefore, the general form of the survivorship curve. Estimates of shape and scale parameters and their confidence intervals can be easily calculated and used to compare survivorship curves of different populations.

Pires, A. M. and J. A. Branco (2010). A statistical model to explain the Mendel-Fisher Controversy. *Statistical Science 25*, 545–565.

In 1866 Gregor Mendel published a seminal paper containing the foundations of modern genetics. In 1936 Ronald Fisher published a statistical analysis of Mendel's data concluding that "the data of most, if not all, of the experiments have been falsified so as to agree closely with Mendel's expectations." This paper provides a probability model that fits Mendel's data and does not offend Fisher's analysis.

Polakow, D. A. and T. T. Dunne (1999). Modelling fire-return interval T: stochasticity and censoring in the two-parameter Weibull model. *Ecological Modelling 121*, 79–102.

It is well understood that the process of fire-return is subject to stochastic variation. However, a deterministic paradigm underlies many contemporary studies of fire-frequency to the possible detriment of robust ecological description. It is also commonplace for data on historical fire processes to contain some degree of either partial or missing data. In this paper, we introduce methods for incorporating parameter stochasticity into the two-parameter Weibull model under different degrees of data censoring.

Ramos, V. and G. Iglesias (2014). Wind power viability on a small island. *International Journal of Green Energy 11*, 741–760.

This work investigates the potential of wind power on the island of Arousa. Rayleigh and Weibull distributions were fitted to annual and monthly wind data. While the Rayleigh distribution tends to overestimate slightly the resource, its Weibull counterpart provides a good fit.

Ricklefs, R. E. (2000). Intrinsic aging-related mortality in birds. *Journal of Avian Biology 31*, 103–111.

Actuarial senescence in captive populations of 28 species of bird was quantified by estimating the parameters of Weibull models fitted to survival curves constructed from data obtained from zoos.

Siipilehto, J. (2009). Modelling stand structure in young Scots pine dominated stands. *Forest Ecology and Management 257*, 223–232.

The purpose of this study was to construct models for predicting the structure of young Scots pine stands. The two-parameter Weibull function characterized the height distribution of the stands. Tree diameters were predicted using a multiplicative model, fitted as a linearized mixed-effect model. The Weibull function was fitted using the maximum likelihood method. Four methods for predicting the distributions were compared; goodness-of-fit were tested in terms of Kolmogorov–Smirnov and error index statistics. Models are recommended as practical tools for Finnish forest management planning purposes.

Son, Y. and E. E. Lewis (2005). Effects of temperature on the reproductive life history of the black vine weevil, *Otiorhynchus sulcatus*. *Entomologia experimentalis et applicata 114*, 15–24.

This study investigates the effects of temperature on the full set of reproductive traits of *O. sulcatus* and develops descriptive models for each trait. The cumulative distribution of oviposition probability through normalized age, i.e., age divided by median longevity, was calculated in terms of percentiles for each temperature. A two-parameter Weibull function (Cockfield et al. (1994)) was used to describe the cumulative oviposition rate against the normalized age as an independent variable.

Symynck, J. (2010). Package 'weibulltoolkit'. http://sourceforge.net/projects/weibulltoolkit.

The 'Weibull Toolkit' for R provides basic functionality for Weibull-based fatigue and reliability analysis. It provides useful plots of the life data, and calculates confidence bounds for B-lives. The toolkit should not (yet) be treated als a viable alternative to better and more reliable software.

Ulgen, K. and A. Hepbasli (2002). Determination of Weibull parameters for wind energy analysis of İzmir, Turkey. *International Journal of Energy Research 26*, 495–506.

In this study, the two Weibull parameters of the wind speed distribution function were computed from the wind speed data for İzmir. The Weibull distribution is found to be suitable to represent the actual probability of wind speed.

Watson, T. G., C. D. Christian, A. J. Mason, M. H. Smith, and R. Meyer (2004). Bayesian-based pipe failure model. *Journal of Hydroinformatics 6*, 259–264.

The overall objective of this research is to develop a Bayesian-based decision support system that will facilitate the identification of efficient asset management policies. An object oriented discrete event simulation has been developed to analyse existing maintenance policies, test the Bayesian methodology and to develop and identify improved maintenance policies. This paper focuses on the areas of research relating to the long term management of water distribution systems.

Watt, M. S., V. Xu, and M. Bloomberg (2010). Development of a hydrothermal time seed germination model which uses the Weibull distribution to describe base water potential. *Ecological Modelling 221*, 1267–1272.

Seed germination has been modelled extensively using hydrothermal time (HTT) models, that predict time to germination as a function of the extent to which seedbed temperature and water potential exceed the base temperature and base water potential of each seed percentile. We fouund that the Weibull distribution more accurately describes both the right skewed distribution of the base water potential for each percentile, and the germination time course over sub-optimal seedbed temperature than the HTT based on the normal distribution.

Yang, Z., Y. X. Chen, Y. F. Li, E. Zio, and R. Kang (2014). Smart electricity meter reliability prediction based on accelerated degradation testing and modeling. *International Journal of Electrical Power and Energy Systems 56*, 209–219.

We conduct accelerated degradation tests for the prediction of smart electricity meter (SEM) reliability. The test data were used to fit degradation paths by linear regression models. Extrapolation to the failure threshold allows the prediction of the

Time-to-Failure of the SEM. Finally, the reliable lifetime of the SEM is predicted by an accelerated degradation function which is obtained by fitting a Weibull failure time distribution.

Zabel, R. W., B. J. Burke, M. L. Moser, and C. C. Caudill (2014). Modeling temporal phenomena in variable environments with parametric models: An application to migrating salmon. *Ecological Modelling 273*, 23–30.

Timing phenomena are integral to many ecological processes but are difficult to analyze due to the unique nature of timing data and because environmental conditions and behavior can vary during the observation period. We developed routines in R to apply parametric models, based on the exponential, Weibull, and modified Weibull distributions, to time-to-event data. We applied the models to data on the time for migrating adult salmonids to successfully pass a hydroelectric dam.

Zobeck, T. M., T. E. Gill, and T. W. Popham (1999). A two-parameter Weibull function to describe airborne dust particle size distributions. *Earth Surface Processes and Landforms 24*, 943–955.

Recent work has shown mathematically how the sequential fragmentation of materials leads to a Weibull distribution. We test the hypothesis that the Weibull distribution may be used to describe airborne soil grains. This study demonstrated that the Weibull cumulative distribution function is an excellent choice to describe the particle size distribution of dust suspended from mineral sediment.

B.3 Parameter estimation for the 2-parameter Weibull distribution

This section summarizes different parameter estimation approaches that have been applied to the 2-parameter Weibull distribution.

Ageel, M. I. (2002). A novel means of estimating quantiles for 2-parameter Weibull distribution under the right random censoring model. *Journal of Computational and Applied Mathematics 149*, 373–380.

Two methods to impute the censored observations are proposed in a right random censoring model for a 2-parameter Weibull distribution. By a Monte Carlo simulation, the quantile estimates are calculated to assess the performance of the proposed imputation methods with respect to their relative mean square error. Simulation results indicate that the two imputation methods proposed are superior to the method proposed by many other authors if the shape parameter of Weibull distribution exceeds 1, except for the lower quantiles.

Altshuler, B. (1989). Quantitative models for lung cancer induced by cigarette smoke. *Environmental Health Perspectives 81*, 107–108.

This discussion paper gives a limited history of work done at the Institute of Environmental Medicine, New York University Medical Center on quantitative modeling relating to lung cancer and cigarette smoking. It examines the proposal that life shortening be considered as a measure of the impact of lung cancer using log normal and Weibull types of distributions.

Balasooriya, U., S. L. C. Saw, and V. Gadag (2000). Progressively censored reliability sampling plans for the Weibull distribution. *Technometrics 42*, 160–167.

This article presents progressively censored variable sampling plans for the Weibull distribution. Approximate maximum likelihood estimators are developed for estimating the parameters of interest. In the construction of sampling plans, asymptotic distribution theory is used to determing the sample size and the acceptance constant. A Monte Carlo experiment has shown that the procedure is sufficiently accurate for practical purposes.

Banerjee, S. and B. P. Carlin (2004). Parametric spatial cure rate models for interval-censored time-to-relapse data. *Biometrics 60*, 268–275.

This article extends existing cure rate models to allow for spatial correlation (estimable via zip code identifiers for the subjects) as well as interval censoring. The approach is Bayesian, where posterior summaries are obtained via a hybrid Markov chain Monte Carlo algorithm.

Banerjee, S., M. M. Wall, and B. P. Carlin (2003). Frailty modeling for spatially correlated survival data, with application to infant mortality in Minnesota. *Biostatistics 4*, 123–142.

In this paper, we consider random effects corresponding to clusters that are spatially arranged, such as clinical sites or geographical regions. Our aim is to explain the pattern of infant mortality using important covariates while accounting for possible differences in hazard among Minnesota counties.

Barmoav, M. F. (2010). Reduced bias factor distribution to reduce the likelihood estimate bias of small sample sizes. In *Reliability and Maintainability Symposium (RAMS), 2010 Proceedings - Annual*, pp. 1–7.

A new method is developed by the author to reduce the bias of the maximum likelihood estimates (MLE) with small sample sizes. The new method is based on a special case of the Fréchet distribution. The new distribution is very flexible and versatile with the scale and shape parameters to fit for relevant scenarios. This function supports any life distributions such as Weibull, Normal, Log Normal and or others distributions as needed.

Bender, R., T. Augustin, and M. Blettner (2005). Generating survival times to simulate Cox proportional hazards models. *Statistics in Medicine 24*, 1713–1723.

Techniques to generate survival times for simulation studies regarding Cox proportional hazards models are presented. A general formula describing the relation between the hazard and the corresponding survival time of the Cox model is derived, which is useful in simulation studies. It is shown how the exponential,the Weibull and the Gompertz distribution can be applied to generate appropriate survival times for simulation studies.

Cacciari, M., G. Mazzanti, G. C. Montanari, and J. Jacquelin (2002). A robust technique for the estimation of the two-parameter Weibull function for complete data sets. *Metron 60*, 65–92.

A statistical procedure is applied to obtain the point estimates of the two parameters of a Weibull distribution function, which is able to describe effectively the experimental results of complete life tests performed on solid dielectric insulation.

Carroll, K. J. (2003). On the use and utility of the Weibull model in the analysis of survival data. *Controlled Clinical Trials 24*, 682–701.

In the analysis of survival data arising in clinical trials, Cox's proportional hazards regression model is firmly established statistical norm, but it makes no assumption about the underlying distribution of survival times. However, if the distribution of survival times can be well approximated, parametric failure-time analyses can be useful, allowing a wider set of inferences to be made. The aim of this paper is to examine the use and utility of the Weibull model in the analysis of survival data from clinical trials and, in doing so, illustrate the practical benefits of a Weibull-based analysis.

Chicheportiche, R. and J.-P. Bouchaud (2012). Weighted Kolmogorov-Smirnov test: accounting for the tails. *Physical Review 86*, 041115.

We have derived exact asymptotic results for a generalization of the Kolmogorov-Smirnov test, well suited to test extreme tails.

Danish, M. Y. and M. Aslam (2014). Bayesian inference for the randomly censored Weibull distribution. *Journal of Statistical Computation and Simulation 84*, 215–230.

We consider the Bayesian inference of the unknown parameters of the randomly censored Weibull distribution.

Dey, D. K. and T.-M. Lee (1992). Bayes computation for life testing and reliability estimation. *IEEE Transactions On Reliability 41*, 621–626.

This paper deals with powerful computational techniques to estimate the parameters and the reliability function of complex life distributions, using Bayes methods, from complete and type-II censored samples. The Gibbs sampler approach brings considerable conceptual and computational simplicity to the calculation of the posterior marginals and reliability. Considering constrained parameter and truncated data problems in multivariate life distributions, the Gibbs sampler procedure is easy to implement for sets of simulated data.

Doostparast, M. (2011). Goodness-of-fit tests for Weibull populations on the basis of records. http://arxiv.org/abs/1110.5509.

In this paper, Kolmogorov-Smirnov and Cramer-von Mises type goodness of fit tests for record data are proposed. Also a new weighted goodness of fit test is suggested. A Monte-Carlo simulation study is conducted to derive the percentiles of the statistics proposed.

dos Santos, D. M., R. B. Davies, and B. Francis (1995). Nonparametric hazard versus nonparametric frailty distribution in modelling recurrence of breast cancer. *Journal of Statistical Planning and Inference 47*, 111–127.

In extending survival models to include frailty effects, the relative merits of parametric and non-parametric formulations are unclear. This paper illustrates and examines some of the issues that arise in applying frailty models to the recurrence of breast cancer. The results suggest marginal advantage for the 'econometric' approach provided that there is no interest per se in the frailty distribution.

Dümbgen, L., K. Rufibach, and D. Schuhmacher (2009). Maximum-likelihood estimation of a log-concave density based on censored data. *Bernoulli 15*, 40–68.

We consider nonparametric maximum-likelihood estimation of a log-concave density in case of interval- or right-censored or binned data. Theoretical properties are studied and an algorithm is proposed for the approximate computation of the estimator.

Erto, P., A. Lanzotti, and A. Lepore (2010). Wind speed parameter estimation from one-month sample via Bayesian approach. *Quality and Reliability Engineering International 26*, 853–862.

This paper proposes to reduce the long-term monitoring of wind data by exploiting other initial information about parameters to be estimated via MCMC (Markov chain Monte Carlo). The proposed Bayesian approach allows the integration of prior information (e.g. obtained from atlases, databases and/or fluid-dynamic assessment) with sampling data, and furnishes effective and timely posterior information about Weibull parameters of the wind speed distribution. The results of the application show that the proposed methodology fits the applicative needs very well. A bootstrap simulation remarks that the attained precision of the Bayesian estimates carried out from a one-month sample is comparable to the maximum likelihood estimates obtained from an actual one-year sample.

Guure, C. B., N. A. Ibrahim, and M. B. Adam (2013). Bayesian inference of the Weibull model based on interval-censored survival data. *Computational and Mathematical Methods in Medicine 2013*, 1–13. Article ID 849520.

We review the traditional maximum likelihood approach to estimating the Weibull parameters with interval-censored data, and also consider the Bayesian approach. The study indicates that the Bayesian estimator is preferred to the classical maximum likelihood estimator for both the scale and shape parameters.

Heitjan, D. F., C. Y. Kim, and H. Li (2004). Bayesian estimation of cost-effectiveness from censored data. *Statistics in Medicine 23*, 1297–1309.

We describe a Bayesian methodology for estimating the cost-effectiveness of a new treatment compared to a standard in a clinical trial, when censoring of survival, the effectiveness variable, induces censoring of total cost. The statistical model assumes that survival follows a Weibull distribution.

Ismail, A. A. (2013). Likelihood inference for a step-stress partially accelerated life test model with Type-I progressively hybrid censored data from Weibull distribution. *Journal of Statistical Computation and Simulation 0*, 1–9.

Progressively hybrid censoring schemes have become quite popular in life testing and reliability studies for high quality products. In this article, the point and interval maximum-likelihood estimations of Weibull distribution parameters and the acceleration factor are considered. The estimation process is performed under Type-I progressively hybrid censored data for a step-stress partially accelerated test model. The biases and mean square errors of the maximum-likelihood estimators are computed to assess their performances in the presence of censoring through a Monte Carlo simulation study.

Jiang, R. (2014). A drawback and an improvement of the classical Weibull probability plot. *Reliability Engineering and System Safety 126*, 135–142.

This paper carries out an analysis for the Weibull transformations that create the WPP plot and shows that the shape of the WPP plot of the data randomly generated from a distribution model can be significantly different from the shape of the WPP plot of the model due to the high non-linearity of the Weibull transformations. A cdf-based weighted least squares method is proposed to improve the parameter estimation accuracy; and an improved WPP plot is suggested to avoid the drawback of the classical WPP plot.

Joly, P., D. Commenges, C. Helmer, and L. Letenneur (2002). A penalized likelihood approach for an illness-death model with interval-censored data: application to age-specific incidence of dementia. *Biostatistics 3*, 433–443.

We consider the problem of estimating the intensity functions for a continuous time 'illness-death' model with intermittently observed data. Estimating the intensity of transition from health to illness by survival analysis (treating death as censoring) is biased downwards. We propose to estimate the intensity functions by maximizing a penalized likelihood. The method yields smooth estimates without parametric assumptions. Simulation data were generated with a mixture of gamma and Weibull distributions.

Jácome, A. A. A., D. R. Wohnrath, C. S. Neto, E. C. Carneseca, S. V. Serrano, L. S. Viana, J. S. Nunes, E. Z. Martinez, and J. S. Santos (2014). Prognostic value of epidermal growth factor receptors in gastric cancer: a survival analysis by Weibull model incorporating long-term survivors. *Gastric Cancer 17*, 76–86.

Tissues from gastric tumors were analyzed. Correlations between receptor expression and clinicopathological characteristics were performed according to the chi-square test. Survival analysis was calculated according to the Weibull model with a mixture model incorporating long-term survivors. Multivariate analysis of prognostic factors was performed by a regression model incorporating long-term survivors with the Weibull distribution.

Kachman, S. D. (1999). Applications in survival analysis. *Journal of Animal Science 2*, 147–153.

This paper provides a brief introduction to survival analysis. Survival analysis provides a set of distributions appropriate for time until event traits. In addition, estimation procedures are equipped to handle various forms of censoring. The Weibull survival model with the addition of time-dependent covariates can handle a wide variety of survival traits. Time-dependent covariates also allow the modeling of events which have an effect of limited duration.

Kaminskiy, M. P. and V. V. Krivtsov (2005). A simple procedure for Bayesian estimation of the Weibull distribution. *IEEE Transactions on Reliability 54*, 612–616.

This paper presents a new procedure for the Bayesian estimation of the Weibull distribution. The novelty of the suggested procedure is that the prior information can be presented in the form of the interval assessment of the reliability function (as opposed to that on the Weibull parameters), which is generally easier to obtain. It also introduces a new parametric form of the prior distribution for the scale parameterof the exponential distribution. This distribution is not a Gamma (as might intuitively be expected); its mode is available in a closed form, and the mean is obtainable through a series approximation.

Kwon, S.-D. (2010). Uncertainty analysis of wind energy potential assessment. *Applied Energy 87*, 856–865.

This study presents a framework to assess the wind resource of a wind turbine using uncertainty analysis. The probability models used include mean wind velocity and associated Weibull parameters.

Lee, E. T. and O. T. Go (1997). Survival analysis in public health research. *Annual review of public health 18*, 105–134.

This paper reviews the common statistical techniques employed to analyze survival data in public health research. Due to the presence of censoring, the data are not amenable to the usual method of analysis. Hazard functions for the exponential, Weibull, gamma, Gompertz, lognormal, and log-logistic distributions are described. The paper is intended for public health professionals who are interested in survival data analysis.

Liang, L.-J., D. Huang, M.-L. Brecht, and Y.-I. Hser (2010). Differences in mortality among heroin, cocaine, and methamphetamine users: a hierarchical Bayesian approach. *Journal of Drug Issues 40*, 121–140.

We introduce a Bayesian framework that jointly models survival data using a Weibull proportional hazard model with frailty, and substance and alcohol data using mixed-effects models, to examine differences in mortality among heroin, cocaine, and methamphetamine users.

Liao, M. and T. Shimokawa (1999). A new goodness-of-fit test for Type-I extreme-value and 2-parameter Weibull distributions with estimated parameters. *Journal of Statistical Computation and Simulation 64*, 23–48.

On the basis of the Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling statistics, a new statistic is developed and applied for testing the goodness-of-fit of Type-I extreme-value and 2-parameter Weibull distributions with estimated parameters. Maximum likelihood estimators and graphical plotting techniques are used to estimate the population parameters from a complete sample. The critical values of the new statistic are calculated using Monte Carlo simulation.

Lu, J.-C. (1992). Bayes parameter estimation for the bivariate Weibull model of Marshall-Olkin for censored data. *IEEE Transactions on Reliability 41*, 608–615.

This article develops a Bayes parameter estimation method for bivariate censored data collected from both system and component levels. The bivariate Weibull distribution of Marshall & Olkin is used to model lifetimes of components in a 2-component system. Closed-form Bayes estimators of the model parameters are proposed along with their variances and high probability density intervals.

Ma, Z. S. and A. W. Krings (2008). Survival analysis approach to reliability, survivability and prognostics and health management (PHM). In *Aerospace Conference, 2008 IEEE*.

In this paper, we attempt to demonstrate, by reviewing and comparing the major mathematical models of survival analysis and reliability theory, that these two fields essentially address the same mathematical problems.

Manton, K. G., E. Stallard, and J. W. Vaupel (1986). Alternative models for the heterogeneity of mortality risks among the aged. *Journal of the American Statistical Association 81*, 635–644.

This paper uses published Medicare mortality rates and national vital statistics data to examine the sensitivity to choice of heterogeneity distribution and hazard rate function.

Meeker, W. Q. and L. A. Escobar (1998). *Statistical Methods for Reliability Data*, Chapter 6. Probability Plotting, pp. 122–151. John Wiley & Sons.

This chapter explains: applications of probability plots; basic probability plotting concepts for both complete and censored data; how to analytically linearize a cdf on special plotting scales; how to plot a modified nonparametric estimate of $F(t)$ on probability paper and how to use such a plot to judge the adequacy of a particular parametric distribution; analytical and simulation methods of separating useful information from "noise" when using a probability plot to assess the reasonableness of a particular distributional model; graphical estimates of important reliability characteristics like failure probabilities and distribution quantiles.

Morris, C. and C. Christiansen (1995). Fitting Weibull duration models with random effects. *Lifetime Data Analysis 1*, 347–359.

Duration time models often should include correlated failure times, due to clustered data. The data analyzed here involve such correlations because patient level outcomes (the times until graft failure following kidney transplantation) are observed, but patients are clustered in different transplant centers. This paper shows how to estimate a Weibull frailty survival model by using back-and-forth iterations between two standard software programs: one for Weibull survival models; and the other for hierarchical Poisson regression.

Nelson, W. (1985). Weibull analysis of reliability data with few or no failures. *Journal of Quality Control 17*, 140–146.

This expository paper presents estimates and confidence limits for Weibull percentiles and reliabilities when the Weibull shape parameter is assumed to have a known value. The methods are particularly useful when there are few or no failures, and a shape parameter value from similar data is available.

Niola, V., R. Oliviero, and G. Quaremba (2005). The application of wavelet transform for estimating the shape parameter of a Weibull pdf. In *WSEAS International Conference on Dynamical Systems and Control*, pp. 126–130.

In this paper a method for the parameter estimation of a Weibull distribution is explained. It is based on the estimates of the cumulative probability function performed by a wavelet method. The estimates, obtained by the application of the proposed method, are compared with the results based on other literature estimators. In particular, it is shown that the estimates are stable also when samples show outliers.

Nordman, D. J. and W. Q. Meeker (2002). Weibull prediction intervals for a future number of failures. *Technometrics, 44*, 15–23.

This paper evaluates exact coverage probabilities of approximate prediction intervals for the number of failures that will be observed in a future inspection of a sample of units, based only on the results of the first in-service inspection of the sample. The failure-time of such units is modeled with a Weibull distribution having a given shape parameter value.

Odell, P. M., K. M. Anderson, and R. B. D'Agostino (1992). Maximum likelihood estimation for interval-censored data using a Weibull-based accelerated failure time model. *Biometrics 48*, 951–959.

The accelerated failure time regression model is most commonly used with right-censored survival data. This report studies the use of a Weibull-based accelerated failure time regression model when left- and interval-censored data are also observed. Simulation studies indicate that for relatively large samples there are many instances when the MLE is superior to the midpoint estimator (MDE). For samples where the hazard rate is flat or nearly so, or where the percentage of interval-censored data is small, the MDE is adequate.

Parsi, S., M. Ganjali, and N. S. Farsipour (2011). Conditional maximum likelihood and interval estimation for two Weibull populations under joint Type-II progressive censoring. *Communications in Statistics - Theory and Methods 40*, 2117–2135.

This study considers the interval estimation for two Weibull populations when joint Type-II progressive censoring is implemented. We obtain the conditional maximum likelihood estimators of the two Weibull parameters under this scheme. Moreover, simultaneous approximate confidence region based on the asymptotic normality of the maximum likelihood estimators are also discussed and compared with two Bootstrap confidence regions.

Pasha, G. R., M. S. Khan, and A. H. Pasha (2006). Discrimination between Weibull and lognormal distributions for lifetime data. *Journal of Research (Science) 17*, 103–114.

The aim of this research is to discriminate between the Weibull and Lognormal distributions for complete samples. Both reliability models Weibull and Lognormal are then illustrated. Median rank regression (MRR) and maximum likelihood estimation (MLE) data- fitting methods are described and goodness-of-fit using maximum likelihood ratio (MLR) and most powerful invariant (MPI) tests. We find the Weibull distribution better for fitting to lifetime data when comparing with the Lognormal distribution.

Prakash, G. and D. Singh (2009). A Bayesian shrinkage approach in Weibull Type-II censored data using prior point information. *REVSTAT 7*, 171–187.

In the present paper we study the performance of the Bayes Shrinkage estimators for the scale parameter of the Weibull distribution under the squared error loss and the LINEX loss functions in the presence of a prior point information of the scale parameter when Type-II censored data are available. The properties of the minimax estimators are also discussed.

Pu, Z. and L. Li (1999). Regression models with arbitrarily interval-censored observations. *Communications in Statistics - Theory and Methods 28*, 1547–1563.

In this paper we study the least squares regression and Cox regression models when data are subject to arbitrary interval censoring. Simulations and an application of a set of cancer treatment data are studied for investigating the performance of the proposed methods.

Saleh, H., A. A. E.-A. Aly, and S. Abdel-Hady (2012). Assessment of different methods used to estimate Weibull distribution parameters for wind speed in Zafarana wind farm, Suez Gulf, Egypt. *Energy 44*, 710–719.

The wind speed data for the Zafarana Project have been analyzed to estimate the most appropriate method to find Weibull distribution parameters for this site. The results from various methods were compared with the data to find their accuracy based on the root mean square errors.

Schuckers, M. E. (2002). Interval estimates when no failures are observed. In *AutoID'02 Proceedings: Workshop on Automatic Identification Advanced Technologies*, pp. 37–41.

In this paper we discuss ways to use Bayesian methodology to estimate the matching performance of a biometric identification device when no errors are detected. One of the drawbacks to the classical or frequentist statistical estimation methods is that it is not possible to create a confidence interval for the error rate when no errors are observed.

Schütz, N. and M. Holschneider (2011). Detection of trend changes in time series using bayesian inference. *Physical Review E 84*, 021120.

Change points in time series are perceived as isolated singularities where two regular trends of a given signal do not match. The detection of such transitions is of fundamental interest for the understanding of the system's internal dynamics or external forcings. In practice observational noise makes it difficult to detect such change points in time series. In this work we elaborate on a Bayesian algorithm to estimate the location of the singularities and to quantify their credibility. We validate the performance and sensitivity of our inference method by estimating change points of synthetic data sets.

Seguro, J. and T. Lambert (2000). Modern estimation of the parameters of the Weibull wind speed distribution for wind energy analysis. *Journal of Wind Engineering and Industrial Aerodynamics 85*, 75–84.

Three methods for calculating the parameters of the Weibull wind speed distribution for wind energy analysis are presented: the maximum likelihood method, the proposed modified maximum likelihood method, and the commonly used graphical method. The application of each method is demonstrated using a sample wind speed data set, and a comparison of the accuracy of each method is also performed.

Singh, S. K., U. Singh, and D. Kumar (2013). Bayesian estimation of parameters of inverse Weibull distribution. *Journal of Applied Statistics 40*, 1597–1607.

The present paper describes the Bayes estimators of parameters of inverse Weibull distribution for complete, type I and type II censored samples under general entropy and squared error loss functions. The proposed estimators have been compared on the basis of their simulated risks (average loss over sample space).

Somboonsavatdee, A., V. N. Nair, and A. Sen (2007). Graphical estimators from probability plots with right-censored data. *Technometrics 49*, 420–429.

We investigate the properties of graphical estimators with multiply right-censored data and compare their performance with that of maximum likelihood estimators (MLEs). The relative errors of these graphical estimators compared with the MLEs suggest that they do poorly for Weibull distributions.

Tjørve, E. (2003). Shapes and functions of species-area curves: A review of possible models. *Journal of Biogeography 30*, 827–835.

This paper reviews possible candidate models that may be used in theoretical modelling and empirical studies of species-area relationships (SARs).

Viveros, R. and N. Balakrishnan (1994). Interval estimation of parameters of life from progressively censored data. *Technometrics 36*, 84–91.

A conditional method of inference is used to derive exact confidence intervals for several life characteristics such as location, scale, quantiles, and reliability when the data are Type II progressively censored.

Wang, W. (2004). Refined rank regression method with censors. *Quality and Reliability Engineering International 20*, 667–678.

The rank regression method with an approach introduced by Johnson has been commonly used to handle data with suspensions in engineering practice and commercial software. However, the Johnson method makes partial use of suspension information only—the positions of suspensions, not the exact times to suspensions. A new approach for rank regression with censored data is proposed in this paper, which makes full use of suspension information.

Wu, D., J. Zhou, and Y. Li (2006). Unbiased estimation of Weibull parameters with the linear regression method. *Journal of the European Ceramic Society 26*, 1099–1105.

Monte Carlo simulations were used to search for the probability estimator for the unbiased estimate of the Weibull parameters in the linear regression method. Compared with commonly-used probability estimators, the estimator proposed gives a more accurate estimation of the Weibull modulus and the same estimation precision of the scale parameter.

Wu, M., Y. Shi, and Y. Sun (2014). Inference for accelerated competing failure models from Weibull distribution under Type-I progressive hybrid censoring. *Journal of Computational and Applied Mathematics 263*, 423–431.

This paper considers constant-stress accelerated competing failure models under Type-I progressive hybrid censoring with binomial random removals. A Weibull distributed life of test units is assumed for a specific cause and by the Newton-Raphson iteration and asymptotic likelihood theory, the maximum likelihood estimates (MLEs) and asymptotic confidence intervals of the unknown parameters are obtained. Based on the noninformative prior, a Gibbs sampling algorithm using adaptive rejection sampling is presented to obtain Bayesian estimates and the Monte Carlo (MC) method is employed to construct the HPD credible intervals. The simulation results are provided to show that Bayesian estimates perform better than MLEs and the change of the removal probability has a significant effect on MLEs.

Wu, S. (2002). Estimations of the parameters of the Weibull distribution with progressively censored data. *Journal of the Japan Statistical Society 32*, 155–163.

We obtained estimation results concerning a progressively type-II censored sample from a two-parameter Weibull distribution. The maximum likelihood method is used to derive the point estimators of the parameters. An exact confidence interval and an exact joint confidence region for the parameters are constructed.

Xiang, L. and S. K. Tse (2005). Maximum likelihood estimation in survival studies under progressive interval censoring with random removals. *Journal of Biopharmaceutical Statistics 15*, 981–991.

Censoring occurs commonly in clinical trials. This article investigates a new censoring scheme, namely, Type II progressive interval censoring with random removals to cope with the setting that patients are examined at fixed regular intervals and dropouts may occur during the study period. We discuss the maximum likelihood estimation of the model parameters and derive the corresponding asymptotic variances when survival times are assumed to be Weibull distributed.

Yang, J. and D. W. Scott (2013a). Case studies of Weibull model fittings.

R code for case studies to support Yang and Scott (2013b)

Yang, J. and D. W. Scott (2013b). Robust fitting of a Weibull model with optional censoring. *Computational Statistics and Data Analysis 67*, 149–161.

A new robust procedure is introduced to fit a Weibull model by using L_2 distance, i.e. integrated square distance, of the Weibull probability density function. The Weibull model is augmented with a weight parameter to robustly deal with contaminated data. It is shown that this new L_2 parametric estimation method is more robust and does a better job than maximum likelihood in the newly proposed Weibull model when data are contaminated. The same preference for L_2 distance criterion and the new Weibull model also happens for right-censored data with contamination.

Zhang, C. W., T. Zhang, D. Xu, and M. Xie (2013). Analyzing highly censored reliability data without exact failure times: an efficient tool for practitioners. *Quality Engineering 25*, 392–400.

The unavailability of exact failure times has compelled us to develop a simplified version of the maximum likelihood estimation for reliability parameters and measures. The approach is applicable to single parameter reliability models including the exponential reliability model and the Weibull reliability model with an assumed or known shape parameter value. In the latter case, it takes advantage of the fact that in many practical situations a reasonable estimate of the Weibull shape parameter is attainable by certain means.

Zhang, Q., C. Hua, and G. Xu (2014). A mixture Weibull proportional hazard model for mechanical system failure prediction utilising lifetime and monitoring data. *Mechanical Systems and Signal Processing 43*, 103–112.

It is sometimes necessary to combine multiple failure modes when analysing the failure of an overall system. In this paper, a mixture Weibull proportional hazard model is proposed to predict the failure of a mechanical system with multiple failure modes. Results show that this mixed model is greatly superior in system failure prediction to the traditional Weibull proportional hazard model.

Zhu, Y., E. Yashchin, and J. R. M. Hosking (2014). Parametric estimation for window censored recurrence data. *Technometrics 56*, 55–66.

For statistical inference from window censored recurrence data, we derive the likelihood function for a model in which the distributions of inter-recurrence intervals in a single path need not be identical and may be associated with covariate information. We assume independence among different sample paths. We propose a distribution to model the effect of external interventions on recurrence processes.

Zuev, K. M., J. L. Beck, S.-K. Au, and L. S. Katafygiotis (2012). Bayesian post-processor and other enhancements of subset simulation for estimating failure probabilities in high dimensions. *Computers and Structures 92–93*, 283–296.

This paper focuses on enhancements to Subset Simulation (SS), proposed by Au and Beck, which provides an efficient algorithm based on MCMC (Markov chain Monte Carlo) simulation for computing small failure probabilities for general high-dimensional reliability problems.

B.4 Simulation experiments testing the statistical properties of estimators

From these papers:

1. what scenarios did they use to guide the experiments?
2. what metrics did they use to summarize the outcomes? (e.g., bias of parameter estimates, MIAE, MISE)
3. how did the compared methods perform under these different metrics?

Ahmed, A. O. M., N. A. Ibrahim, M. B. Adam, and J. Arasan (2012). Bayesian survival and hazard estimate for Weibull censored time distribution. *Journal of Applied Sciences 12*, 1313–1317.

We use a Bayesian paradigm for Weibull parameter estimation, considering the Jeffreys and extension of Jeffreys prior with the squared loss function. The Bayes estimates of the survival function and hazard rate of the Weibull distribution with censored data obtained using Lindley's approximation are then compared to its maximum likelihood counterparts. The comparison criterion is the Mean Square Error (MSE).

Ateya, S. F. (2013). Estimation under modified Weibull distribution based on right censored generalized order statistics. *Journal of Applied Statistics 40*, 2720–2734.

In this paper, the maximum likelihood (ML) and Bayes methods are considered via Markov chain Monte Carlo to estimate the parameters of three-parameter modified Weibull distribution based on a right censored sample of generalized order statistics. Some comparisons are carried out between the ML and Bayes methods by computing the mean squared errors (MSEs), Akaike's information criteria (AIC) and Bayesian information criteria (BIC) of the estimates.

Bar-Lev, S. K. (2004). Likelihood-based inference for the shape parameter of a two parameter Weibull distribution. *Lifetime Data Analysis 10*, 293–308.

Based on a Type 2 censored sample, we use the likelihood-based approach to draw likelihood inference on the shape parameter gamma of a two-parameter Weibull distribution.

Barbiero, A. (2013). Parameter estimation for type III discrete Weibull distribution: A comparative study. *Journal of Probability and Statistics 2013*, 946562.

This paper describes three methods for estimating the parameters of the type III Weibull distribution: the method of proportion, the ML and the method of moments. The techniques' peculiarities and practical limits are outlined, extensive Monte Carlo simulations performed, and compared in terms of bias and RMSE.

Cacciari, M., G. Mazzanti, and G. C. Montanari (1996). Comparison of maximum likelihood unbiasing methods for the estimation of the Weibull parameters. *IEEE Transactions on Dielectrics and Electrical Insulation 3*, 18–27.

The technique of unbiasing the maximum likelihood estimates of the scale and shape parameters of the Weibull function is discussed. The efficiency of recent and traditional unbiasing estimators is compared, in particular, the Bain-Engelhardt, Harter & Moore and Ross unbiasing methods, and the Jacquelin estimators.

Chu, Y.-K. and J.-C. Ke (2012). Computation approaches for parameter estimation of Weibull distribution. *Mathematical and Computational Applications 17*, 39–47.

This paper examines the estimation comparison of two methods for Weibull parameters, one is the maximum likelihood method and the other is the least squares method. A numerical simulation study is carried out to understand performance of the two methods. Based on sample root mean square errors, we make a comparison between the two computation approaches. We find that the least squares method significantly outperforms the maximum likelihood when the sample size is small.

Denecke, L. and C. H. Müller (2014). New robust tests for the parameters of the Weibull distribution for complete and censored data. *Metrika 77*, 585–607.

Using the likelihood depth, new consistent and robust tests for the parameters of the Weibull distribution are developed. Uncensored as well as type-I right-censored data are considered. Tests are given for the shape parameter and also the scale parameter of the Weibull distribution, where in each case the situation that the other parameter is known as well the situation that both parameter are unknown is examined. The new tests based on likelihood depth are comparable to standard methods and robust against contamination.

Elmahdy, E. E. and A. W. Aboutahoun (2013). A new approach for parameter estimation of finite Weibull mixture distributions for reliability modeling. *Applied Mathematical Modelling 37*, 1800–1810.

The aim of this paper is to model lifetime data for systems that have failure modes by using the finite mixture of Weibull distributions. The proposed approach uses different methods to develop the estimates such as MLE through the EM algorithm. In addition, Bayesian estimations are investigated, and some other extensions such as Graphic, Non-Linear Median Rank Regression and Monte Carlo simulation methods can be used.

Erto, P. (1982). New practical Bayes estimators for the 2-parameter Weibull distribution. *IEEE Transactions on Reliability 31*, 194–197.

This paper directly incorporates prior information on the range of the shape parameter and the anticipated value of a quantile into a Bayesian estimation process, using a new (not completely specified) prior distribution. A Monte Carlo simulation shows that these estimators are quite s-unbiased and s-efficient for a large range of parameter values of poor priors.

Faucher, B. and W. Tyson (1988). On the determination of Weibull parameters. *Journal of Materials Science Letters 7*, 1199–1203.

Determines the correct weighting factors that should be used for determining Weibull parameters in a linear regression.

Fernández, A. and M. Vázquez (2012). Improved estimation of Weibull parameters considering unreliability uncertainties. *IEEE Transactions on Reliability 61*, 32–40.

We propose a linear regression method for estimating Weibull parameters from life tests. The method uses stochastic models of the unreliability at each failure instant. As a result, a heteroscedastic regression problem arises that is solved by weighted least squares minimization.

Genschel, U. and W. Q. Meeker (2010). A comparison of maximum likelihood and median-rank regression for Weibull estimation. *Quality Engineering 22*, 236–255.

This paper reports on the results of a simulation study, to provide insight into the differences between the competing methods of maximum likelihood and median ranked regression.

Gibbons, D. I. and L. C. Vance (1981). A simulation study of estimators for the 2-parameter Weibull distribution. *IEEE Transactions on Reliability 30*, 61–66.

Seven estimators for the scale and shape parameters and percentiles of the Weibull distribution were compared by Monte Carlo methods. The evaluated estimators include the maximum likelihood estimator, linear estimators, least squares estimators and a moment estimator. The performance of these with respect to mean square error was studied in complete and Type II censored samples.

Guure, C. B. and N. A. Ibrahim (2013). Methods for estimating the 2-parameter Weibull distribution with Type-I censored data. *Research Journal of Applied Sciences, Engineering and Technology 5*, 689–694.

We investigate whether Rank Regression Method can be a good alternative to Maximum Likelihood for estimating two parameters of the Weibull distribution. The methods considered are: Maximum Likelihood Estimation, Least Square Estimation on Y and that of Least Square Estimation on X. These estimators are derived for Random Type-I censored samples. These methods were compared using Mean Square Error and Mean Percentage Error through simulation study with small, medium and large sample sizes in estimating the Weibull parameters under Type-I censored data.

Hirose, H. (1999). Bias correction for the maximum likelihood estimates in the two-parameter Weibull distribution. *IEEE Transactions on Dielectrics and Electrical Insulation 6*, 66–68.

This paper provides a simple bias correction method for the maximum likelihood estimates in the 2-parameter Weibull distribution. The unbiased estimates of the shape and scale parameters as well as the percentile points can be obtained by using a simple formula.

Hong, Y., W. Q. Meeker, and L. A. Escobar (2008). The relationship between confidence intervals for failure probabilities and life time quantiles. *IEEE Transactions on Reliability 57*, 260–266.

We compare s-normal approximation to likelihood methods, and introduce a new procedure to get the confidence intervals for the failure probability function by inverting the pointwise confidence bands of the quantile function.

Hossain, A. and W. Zimmer (2003). Comparison of estimation methods for Weibull parameters: Complete and censored samples. *Journal of Statistical Computation and Simulation 73*, 145–153.

This paper compares several methods for estimating the parameters of the two-parameter Weibull distribution with complete, multiply time censored, and type II censored samples. An extensive simulation study compares the performance of these estimators. The maximum likelihood and least square methods are compared using the bias and the mean squared error (MSE) of the estimates.

Jeng, S.-L. and W. Q. Meeker (2000). Comparisons of approximate confidence interval procedures for Type I censored data. *Technometrics 42*, 135–148.

This article compares different procedures to compute confidence intervals for parameters and quantiles of the Weibull, lognormal and similar log-locaation-scale distributions from Type I censored data.

Khalili, A. and K. Kromp (1991). Statistical properties of Weibull estimators. *Journal of Materials Science 26*, 6741–6752.

Weibull parameters were estimated for data produced by Monte Carlo simulations using three different approaches: linear regression, moments method, and maximum likelihood method.

Kuchii, S., N. Kaio, and S. Osaki (1979). Simulation comparisons of point estimation methods in the 2-parameter Weibull distribution. *Microelectronics Reliability 19*, 333–336.

The following three estimation methods in the 2-parameter Weibull distribution are well-known: a method of maximum-likelihood estimation, a method of coefficient of variation, and a method of Weibull probability paper. By simulation we discuss which of these methods is better and conclude that a method of Weibull probability paper is relatively better both for complete samples and Type I censored samples.

Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics 50*, 144–154.

This article deals with the Bayesian inference of unknown parameters of the progressively censored Weibull distribution. We use Lindley's approximation to compute the Bayes estimates and the Gibbs sampling procedure to calculate the credible intervals. For given priors, we also provide a methodology to compare two different censoring schemes and thus find the optimal Bayesian censoring scheme. Monte Carlo simulations are performed to observe the behavior of the proposed methods, and a data analysis is conducted for illustrative purposes.

Lawless, J. F. (2010). Discussion of the articles by Olteanu and Freeman, and Genschel and Meeker. *Quality Engineering 22*, 278–280.

Overview and summary of Genschel and Meeker (2010) (*q.v.*) and Olteanu and Freeman (2010) (*q.v.*).

Mahmoud, M. R. and R. M. Mandouh (2012). Maximum likelihood estimation of two unknown parameter of Beta-Weibull distribution under Type II censored samples. *Applied Mathematical Sciences 6*, 2369–2384.

In this paper, the maximum likelihood estimates are obtained for the two unknown parameters of the Beta-Weibull distribution under type II censored samples. For each simulated sample the absolute relative biases, mean square errors, relative mean square errors and relative root mean square errors are computed.

McCool, J. I. (1970). Inference on Weibull percentiles and shape parameter from maximum likelihood estimates. *IEEE Transactions on Reliability 19*, 2–9.

The purpose of this paper is to show how ML estimates may be used to set exact confidence limits on and compute median unbiased estimates of the Weibull shape parameter, a Weibull percentile, the ratio of the Weibull shape parameters in two populations, and the ratio of a specific Weibull percentile in two populations having a common but unknown shape parameter.

Montanari, G. C., G. Mazzanti, M. Cacciari, and J. C. Fothergill (1997a). In search of convenient techniques for reducing bias in the estimation of Weibull parameters for uncensored tests. *IEEE Transactions on Dielectrics and Electrical Insulation 4*, 306–313.

Techniques for estimating the parameters of the 2-parameter Weibull distribution from data obtained from uncensored tests are compared. It is shown that common techniques, such as least squares regression and maximum likelihood, may give rise to very significant errors in terms of the bias of the estimated Weibull parameters.

Montanari, G. C., G. Mazzanti, M. Cacciari, and J. C. Fothergill (1997b). Optimum estimators for the Weibull distribution of censored data: Singly-censored tests. *IEEE Transactions on Dielectrics and Electrical Insulation 4*, 462–469.

Six techniques (maximum likelihood, least squares regression and the Jacquelin, Ross, White and Bain Engelhardt estimators) are compared in terms of their simplicity and accuracy in estimating the shape and scale parameters of the 2-parameter Weibull distribution applied to singly censored data.

Montanari, G. C., G. Mazzanti, M. Cacciari, and J. C. Fothergill (1998). Optimum estimators for the Weibull distribution from censored test data: Progressively-censored tests. *IEEE Transactions on Dielectrics and Electrical Insulation 5*, 157–164.

Nair, V. N. (1984). On the behavior of some estimators from probability plots. *Journal of the American Statistical Association 79*, 823–831.

In this paper, estimators from weighted least squares are considered, and their asymptotic, finite-sample, robustness and optimality properties are discussed. Included among these are the ordinary least squares estimators and estimators from least squares lines fitted after trimming or Winsorizing some of the extreme order statistics.

Ng, H. K. T. and Z. Wang (2009). Statistical estimation for the parameters of Weibull distribution based on progressively type-I interval censored sample. *Journal of Statistical Computation and Simulation 79*, 145–159.

Estimation of parameters based on a progressively type-I interval censored sample from a two-parameter Weibull distribution is studied. Different methods of estimation are discussed. They include the maximum likelihood estimation, method of moments, estimation based on Weibull probability plot, mid-point approximation method and one-step approximation method. The estimation procedures are discussed in details and compared via Monte Carlo simulations in terms of their biases and mean square errors.

Olteanu, D. and L. Freeman (2010). The evaluation of median-rank regression and maximum likelihood estimation techniques for a two-parameter Weibull distribution. *Quality Engineering 22*, 256–272.

This study seeks to evaluate maximum likelihood estimation (MLE) and median-rank regression (MRR) for small numbers of failures and high degrees of censoring, where one cannot depend on the asymptotic properties of maximum likelihood estimation, and for extremely high censoring cases, focusing on censoring greater than 99%. We compare the two estimation methods based on parameter estimation and ability to predict future failures.

Peterlik, H. (1995). The validity of Weibull estimators. *Journal of Materials Science 30*, 1972–1976.

The parameters of the two-parametric Weibull distribution, the Weibull modulus and the scale parameter, were estimated by using not only analytical means but also Monte-Carlo simulations. It is shown that the variation coefficient of the scale parameter is dependent on the number of experiments, *M*, which were performed, and on the Weibull modulus itself, whereas the variation coefficient of the Weibull modulus is only dependent on *M*.

Pobočíková, I. and Z. Sedliačková (2012). The least square and the weighted least square methods for estimating the Weibull distribution parameters: A comparative study. *Komunikácie 14*, 88–93.

This paper examines the performance of the least square method and the weighted least square method for estimating the Weibull distribution parameters. Methods are compared in terms of the root mean square error and sample size.

Quigley, J. and M. Revie (2011). Estimating the probability of rare events: Addressing zero failure data. *Risk Analysis 31*, 1120–1132.

We propose estimating the probability of an event occurring through minimax inference on the probability that future samples of equal size realize no more events that that in the data on which the inference is based. Although motivated by inference on rare events, the method is not restricted to zero event data and closely approximates the maximum likelihood estimate (MLE) for nonzero data. A comparison is made with the MLE and regions of the underlying probability are identified where this approach is superior.

Razali, A. M. and A. A. Al-Wakeel (2013). Mixture Weibull distributions for fitting failure times data. *Applied Mathematics and Computation 219*, 11358–11364.

In this paper a mixture of two and three Weibull distributions were used to analyze the data of failure times. The suitability of the distributions is judged from various tests-of-fit. Maximum likelihood estimation MLE was used to estimate the parameters. We found that two- and three-component mixture Weibull distribution provides suitable fits for the failure time data studied based on the shapes of density and hazard functions.

Razali, A. M., A. A. Salih, and A. A. Mahdi (2009). Estimation accuracy of Weibull distribution parameters. *Journal of Applied Sciences Research 5*, 790–795.

This paper compares three methods for estimating the parameters of Weibull distributions: moments, maximum likelihood and least squares. We generated a set of data for the 2-parameter Weibull distribution, and another set for the 3-parameter Weibull distribution and we used these methods to estimate the parameters. We used the means square error and total deviation as measurement for the comparison. We find that the moments method is the best for estimating the parameters because they give the least value for the mean square error.

Skinner, K. R., J. B. Keats, and W. J. Zimmer (2001). A comparison of three estimators of the Weibull parameters. *Quality and Reliability Engineering International 17*, 249–256.

Using mean square error as the criterion, we compare two least squares estimates of the Weibull parameters based on non-parametric estimates of the unreliability with the maximum likelihood estimates (MLEs). The two nonparametric estimators are that of Herd–Johnson and one recently proposed by Zimmer.

Teimouri, M., S. M. Hoseini, and S. Nadarajah (2013). Comparison of estimation methods for the Weibull distribution. *Statistics 47*, 93–109.

The authors propose an L-moment estimator for the Weibull distribution. Then, a comprehensive comparison is made of the following methods: the method of maximum likelihood estimation (MLE), the method of logarithmic moments, the percentile method, the method of moments and the method of L-moments.

Tsang, A. H. C. and A. K. S. Jardine (1993). Estimators of 2-parameter Weibull distributions from incomplete data with residual lifetimes. *IEEE Transactions on Reliability 42*, 291–298.

The exact ML (maximum likelihood) estimators for dealing with sampled data with residual lifetimes are formulated. Monte Carlo simulation was used to compare the performance of ML estimators for various approaches to the treatment of residual data. Two types of LS (least squares) estimators were also evaluated: LSMR (LS median rank) estimators and LSNPML (LS non-parametric ML) estimators.

van Zyl, J. M. and R. Schall (2012). Parameter estimation through weighted least-squares rank regression with specific reference to the Weibull and Gumbel distributions. *Communications in Statistics - Simulation and Computation 41*, 1654–1666.

Using large sample properties of the empirical distribution function and order statistics, weights to stabilize the variance are derived. Weighted least squares regression is then applied to the estimation of the parameters of the Weibull, and the Gumbel distribution. The weights are independent of the parameters of the distributions considered. Monte Carlo simulation shows that the weighted least-squares estimators outperform the usual least-squares estimators totally, especially in small samples.

Wu, S.-J. and C. Kuş (2009). On estimation based on progressive first-failure-censored sampling. *Computational Statistics and Data Analysis 53*, 3659–3670.

In this paper, a new life test plan called a progressive first-failure-censoring scheme is introduced. Maximum likelihood estimates, exact and approximate confidence intervals and an exact confidence region for the parameters of the Weibull distribution are discussed for the new censoring scheme.

Yahgmaei, F., M. Babanezhad, and O. S. Moghadam (2013). Bayesian estimation of the scale parameter of inverse Weibull distribution under the asymmetric loss functions. *Journal of Probability and Statistics 2013*, 1–8. Article ID 890914.

This paper proposes different methods of estimating the scale parameter in the inverse Weibull distribution (IWD). Specifically, the maximum likelihood estimator of the scale parameter in IWD is introduced. We then derived the Bayes estimators for the scale parameter in IWD by considering quasi, gamma, and uniform priors distributions under the square error, entropy, and precautionary loss functions. Finally, the different proposed estimators have been compared by the extensive simulation studies in corresponding the mean square errors and the evolution of risk functions.

Yavuz, A. A. (2013). Estimation of the shape parameter of the Weibull distribution using linear regression methods: Non-censored samples. *Quality and Reliability Engineering International 29*, 1207–1219.

In this paper, we present various regression methods for estimating the Weibull shape parameter and an experimental study using classical regression methods to compare the results of the methods. The parameter estimators considered are: ordinary least squares, weighted least squares (WLS, Bergman, F&T, Lu), non-parametric robust Theil's and weighted Theil's, robust Winsorized least squares, and M-estimators (Huber, Andrew, Tukey, Cauchy, Welsch, Hampel and Logistic). Estimator performances were compared based on bias and mean square error criteria using Monte-Carlo simulations.

Zhang, L. F., M. Xie, and L. C. Tang (2006). Bias correction for the least squares estimator of Weibull shape parameter with complete and censored data. *Reliability Engineering and System Safety 91*, 930–939.

This paper investigates the methods for bias correction when model parameters are estimated with LSE based on probability plot. A cdf-based weighted least squares method is proposed to improve the parameter estimation accuracy; and an improved WPP plot is suggested to avoid the drawback of the classical WPP plot. The appropriateness and usefulness of the proposed estimation method and probability plot are illustrated by simulation and real-world examples.

Zhang, L. F., M. Xie, and L. C. Tang (2007). A study of two estimation approaches for parameters of Weibull distribution based on WPP. *Reliability Engineering and System Safety 92*, 360–368.

A comparison between the estimators of the two LS regression methods Y on X and X on Y using intensive Monte Carlo simulations.

Zhou, D., C. Li, and Z. Wang (2013). Comparison of parameter estimation methods for transformer Weibull lifetime modelling. *International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering 2*, 1170–1177.

Six popular parameter estimation methods (i.e. the maximum likelihood estimation method, two median rank regression methods including the one regressing X on Y and the other one regressing Y on X, the Kaplan-Meier method, the method based on cumulative hazard plot, and the Li's method) are reviewed and compared in order to find the optimal one that suits transformer's Weibull lifetime modelling.

B.5 Miscellaneous papers

Papers of more general interest.

Almalki, S. J. (2013). A reduced new modified Weibull distribution. http://adsabs.harvard.edu/abs/2013arXiv1307.3925A.

In this paper, we propose a reduced version of the new modified Weibull (NMW) distribution due to Almalki and Yuan in order to avoid some estimation problems. The number of parameters in the NMW distribution is five. The number of parameters in the reduced version is three. We study mathematical properties as well as maximum likelihood estimation of the reduced version.

Almalki, S. J. and S. Nadarajah (2014). Modifications of the Weibull distribution: A review. *Reliability Engineering and System Safety 124*, 32–55.

This paper gives an extensive review of some discrete and continuous versions of the modifications of the Weibull distribution.

Bailey, R. T. (1997). Estimation from zero-failure data. *Risk Analysis 17*, 375–380.

In this paper, a model for predicting the binomial failure probability from data that include no failures is examined. The use of this model is currently limited to risk analysis of energetic initiation in the explosives testing field.

Balakrishnan, N. and R. A. Sandhu (1995). A simple simulational algorithm for generating progressive Type-II censored samples. *The American Statistician 49*, 229–230.

We establish an independence result concerning a progressive Type-II censored sample from the uniform distribution. This result is used to present a simple and efficient simulational algorithm for generating a progressive Type-II censored sample from any continuous distribution.

Bhat, H. S. and N. Kumar (2010). On the derivation of the Bayesian Information Criterion. *School of Natural Sciences, University of California*.

We present a careful derivation of the Bayesian Inference Criterion (BIC) for model selection. The BIC is viewed here as an approximation to the Bayes Factor. One of the main ingredients in the approximation, the use of Laplace's method for approximating integrals, is explained well in the literature. Our derivation sheds light on this and other steps in the derivation, such as the use of a flat prior and the invocation of the weak law of large numbers, that are not often discussed in detail.

Block, A. D. and L. M. Leemis (2008). Parametric model discrimination for heavily censored survival data. *IEEE Transactions On Reliability 57*, 248–259.

We consider a plot of the skewness versus the coefficient of variation for the purpose of discriminating among parametric survival models. We extend the method of Cox & Oakes from complete to censored data by developing an algorithm based on a competing risks model and kernel function estimation.

Canfield, R. V. (1992). Conservative Bayes experimental design. *IEEE Transactions On Reliability 41*, 627–631.

Existing Bayes design theory for acceptance/demonstration tests is heuristically examined in this paper for sensitivity of the design to changes in the prior distribution. The changes considered are associated with location and shape (information content) of the prior.

Cousins, R. D. (2011). Negatively biased relevant subsets induced by the most-powerful one-sided upper confidence limits for a bounded physical parameter. arXiv preprint arXiv:1109.2023.

This paper examines the issues associated with using the one-sided upper confidence limit for a measured value in the context of high energy physics.

Duerinckx, M., C. Ley, and Y. Swan (2014). Maximum likelihood characterization of distributions. *Bernoulli 20*, 775–802.

Gauss' principle states that the maximum likelihood estimator of the parameter in a location family is the sample mean for all samples of all sample sizes if and only if the family is Gaussian. There exist many extensions of this result in diverse directions. In this paper we propose a unified treatment of this literature.

Geyer, C. J. (2003). Maximum likelihood in R. http://www.stat.umn.edu/geyer/5931/mle/mle.pdf.

Hayes, K. R. (2011). Uncertainty and uncertainty analysis methods. Technical report, CSIRO. EP102467.

Issues in quantitative and qualitative risk modeling with application to import risk assessment ACERA project (0705).

Hoeting, J. A., D. Madigan, A. E. Raftery, and C. T. Volinsky (1999). Bayesian model averaging: a tutorial. *Statistical Science 14*, 382–417.

Bayesian model averaging (BMA) provides a coherent mechanism for accounting for model uncertainty. Several methods for implementing BMA have recently emerged. We discuss these methods and present a number of examples.

Hong, Y., H. Ma, and W. Q. Meeker (2010). A tool for evaluating time-varying-stress accelerated life test plans with log-location-scale distributions. *IEEE Transactions on Reliability 59*, 620–627.

This paper presents a new approach for computing approximate large-sample variances of maximum likelihood estimators of a quantile of general a log-location-scale distribution with censoring and time-varying stress. The approach is based on a cumulative exposure model.

Inagaki, T. and Y. Ikebe (1984). Direct computation of expected numbers of failures and repairs via integral equation approach. Technical report, University of Tsukuba.

This paper develops a method for computing the expected number of failures and the expected number of repairs of a component in a prescribed time interval.

Jarner, S. F. and G. O. Roberts (2007). Convergence of heavy-tailed Monte Carlo Markov Chain algorithms. *Scandinavian Journal of Statistics 34*, 781–815.

We show polynomial convergence rates of Monte Carlo Markov Chain algorithms with polynomial target distributions, in particular random-walk Metropolis algorithms, Langevin algorithms and independence samplers. We also use similar methodology to consider polynomial convergence of the Gibbs sampler on a constrained state space.

Jeng, S.-L. and W. Q. Meeker (2001). Parametric simultaneous confidence bands for cumulative distributions from censored data. *Technometrics 43*, 450–461.

This article decribes existing methods and develops new methods for constructing simultaneous confidence bands for a cdf. A simulation for the Weibull distribution and time-censored (Type-I) data shows that bootstrap methods provide coverage probabilities that are closer to nominal that those based on the usual large-sample approximations.

Jordan, S. M. and M. H. Zaman (1999). Using SAS macros to develop confidence intervals for the Weibull and extreme value distribution using Type II censored data. In *Proceedings of the 24th Annual SAS Users Group International Conference.*

This paper presents two methods in SAS for producing confidence intervals, representing two different schools of thought—a conditional method (after Lawless 1972, 1978) and classical method (Monte Carlo).

Liu, C.-C. and P. C. T. Willey (2008). The approximate equations between the Weibull and Lognormal distributions by using MRR. In *International Conference on Business and Information.*

As the Weibull and Lognormal distributions are not from the same mathematical family, we are unable to derive mathematical relationships between their shape and scale parameters directly. This study tries to use a simulation method to determine their approximate relationship.

Loparco, F. and M. N. Mazziotta (2011). A Bayesian approach to evaluate confidence intervals in counting experiments with background. *Nuclear Instruments and Methods in Physics Research A646*, 167–173.

In this paper we propose a procedure to evaluate Bayesian confidence intervals in counting experiments where both signal and background fluctuations are described by the Poisson statistics. The results obtained when the method is applied to the calculation of upper limits are also illustrated.

Miller, K. W., L. J. Morell, R. E. Noonan, S. K. Park, D. M. Nicol, B. W. Murrill, and J. M. Voas (1992). Estimating the probability of failure when testing reveals no failures. *IEEE Transactions on Software Engineering 18*, 33–43.

In this paper we introduce formulae for estimating the probability of failure when testing reveals no errors. The formulae are based on a discrete sample space statistical model of software and include Bayesian prior assumptions.

Olsen, J. R., J. H. Lambert, and Y. Y. Haimes (1998). Risk of extreme events under nonstationary conditions. *Risk Analysis 18*, 497–510.

This paper uses a Gumbel Type I distribution to model the probability of failure under nonstationary conditions. The probability of an extreme event under nonstationary conditions depends on the rate of change of the parameters of the underlying distribution.

Schneider, H. (1989). Failure-censored variables-sampling plans for lognormal and Weibull distributions. *Technometrics 31*, 199–206.

This article discusses the design of variables-sampling plans based on failure-censored samples. The advantage of failure-censored sampling plans is that they require a much smaller sample size than attributes-sampling plans and a greatly reduced test time when compared to complete variables-sampling plans.

Vander Wiel, S. A. and W. Q. Meeker (1990). Accuracy of approx confidence bounds using censored Weibull regression data from accelerated life tests. *IEEE Transactions on Reliability 39*, 346–351.

Accelerated life tests (ALTs) subject test units to higher than usual stresses to estimate the time-to-failure distribution at usual stresses. The approximated confidence bounds arising from the large-sample normal distribution of the ML estimates are crude when there are only a few failures. This paper examines the alternative method of setting confidence bounds based on likelihood ratio tests.

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Seminal work introducing the Weibull distribution.

Zhang, Y. and W. Q. Meeker (2002). Bayesian life test planning for the Weibull distribution with given shape parameter. http://www.stat.iastate.edu/preprint/articles/2002-03.pdf.

This paper describes Bayesian methods for life test planning with Type II censored data from a Weibull distribution, when the Weibull shape parameter is given. We use conjugate prior distributions and criteria based on estimating a quantile of interest of the lifetime distribution.